

The Complexity of Approximately Counting Stable Roommate Assignments

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Abstract

We investigate the complexity of *approximately counting* stable roommate assignments in two models: (i) the k -attribute model, in which the preference lists are determined by dot products of “preference vectors” with “attribute vectors” and (ii) the k -Euclidean model, in which the preference lists are determined by the closeness of the “positions” of the people to their “preferred positions”. Exactly counting the number of assignments is $\#P$ -complete, since Irving and Leather demonstrated $\#P$ -completeness for the special case of the stable marriage problem [12]. We show that counting the number of stable roommate assignments in the k -attribute model ($\#k$ -ATTRIBUTE SR, $k \geq 4$) and the 3-Euclidean model ($\#k$ -EUCLIDEAN SR, $k \geq 3$) is interreducible, in an approximation-preserving sense, with counting independent sets (of all sizes) ($\#IS$) in a graph, or counting the number of satisfying assignments of a Boolean formula ($\#SAT$). This means that there can be no FPRAS for any of these problems unless $NP=RP$. As a consequence, we infer that there is no FPRAS for counting stable roommate assignments ($\#SR$) unless $NP=RP$. Utilizing previous results by the authors [3], we give an approximation-preserving reduction from counting the number of independent sets in a bipartite graph ($\#BIS$) to counting the number of stable roommate assignments both in the 3-attribute model and in the 2-Euclidean model. $\#BIS$ is complete with respect to approximation-preserving reductions in the logically-defined complexity class $\#RHH_1$. Hence, our result shows that an FPRAS for counting stable roommate assignments in the 3-attribute model would give an

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FPRAS for all of $\#RHI_1$. We also show that the 1-attribute stable roommate problem always has either one or two stable roommate assignments, so the number of assignments can be determined exactly in polynomial time.

1 Introduction

The *stable roommate problem* is a generalization of the classical *stable marriage problem*. An instance of the roommate problem consists of $2n$ people, where each person has a strict preference ordering (a total ordering) of the other $2n - 1$ people. A *matching* is a pairing of the people into n pairs, and a matching is said to be *stable* if there does not exist a pair of two people P_1 and P_2 , each of whom prefers the other over their current partners in the matching. Such a pair is referred to as a *blocking pair* as P_1 and P_2 would drop their current partners and pair up together.

The stable marriage problem is the special case in which the $2n$ people consist of n men and n women, and each man ranks all of the women higher than any other man and, similarly, each woman ranks all of the men higher than any other woman. (This is not the usual definition of the stable marriage problem, but is equivalent to the standard one.) In 1962, Gale and Shapley proved that every stable marriage instance has a stable matching, and described an $O(n^2)$ algorithm for finding one [6]. The stable marriage problem, including many variants, has seen much study as algorithms for finding stable matchings are used for assigning residents to hospitals in Scotland, Canada, and the USA [2, 15, 17].

More than twenty years after Gale and Shapley’s seminal paper, Irving provided an efficient algorithm for the stable roommate problem [11]. In contrast to the marriage problem, an instance of the roommate problem need not have any stable matching, as this example (which may be found in both [11] and [14]) demonstrates:

Person	Preference list		
A	B	C	D
B	C	A	D
C	A	B	D
D	arbitrary		

Irving’s polynomial-time algorithm determines whether a stable roommate assignment exists for the given instance, and constructs a stable assignment if one exists.

In what follows, we will abbreviate “stable roommate problem” and “stable marriage problem” as SR and SM, respectively.

Since the problem of determining whether a stable assignment exists is solved in both the stable roommate setting and the stable matching setting, it is natural

to ask whether it is feasible to determine *how many* stable assignments there are for a given instance. We denote these counting versions of SR and SM as $\#SR$ and $\#SM$, respectively.

Irving and Leather [12] demonstrated that $\#SM$ (counting the *number* of stable matchings for a given SM instance) is $\#P$ -complete. This completeness result relies on the connection between stable marriages and *downsets* in a related partial order and on the fact that counting downsets in a partial order is $\#P$ -complete [16]. As $\#SM$ is a restricted version of $\#SR$, we can obviously conclude that $\#SR$ is $\#P$ -complete.

Since exactly counting stable matchings is difficult (under standard complexity-theoretic assumptions), it would be good to have algorithms for *approximately* counting. In particular, we would like to find a *fully-polynomial randomized approximation scheme* (an *FPRAS*) for this task, i.e. an algorithm that provides an arbitrarily close approximation in time polynomial in the input size and the desired error.

Randomized approximation schemes have proven successful (sometimes under certain restrictions or conditions) for problems such as counting the number of (perfect) matchings in bipartite graphs, the number of proper k -colorings of graphs, and the number of linear extensions of a partial order. Many of these approximation schemes rely on the Markov Chain Monte Carlo (MCMC) method. This technique also exploits a relationship between counting and sampling described by Jerrum, Valiant, and Vazirani [13], namely, for *self-reducible* combinatorial structures, the existence of an FPRAS is computationally equivalent to a polynomial-time algorithm for approximate sampling from the set of structures.

Bhatnagar, Greenberg, and Randall [1] considered the problem of sampling a random stable matching for the stable marriage problem using the MCMC method. They examined a natural Markov chain that uses “male-improving” and “female-improving” *rotations* (see Section 3.1 for similar definitions in the context of the roommate problem) to define a random walk on the state space of stable matchings for a given instance. In the most general setting, matching instances can be exhibited for which the *mixing time* of the random walk has an exponential lower bound, meaning that it will take an exponential amount of time to (approximately) sample a random stable matching. This exponential mixing time is due to the existence of a “bad cut” in the state space. Bhatnagar, et al. considered several restricted settings for matching instances and were still able to show instances for which such a bad cut exists in the state space, implying an exponential mixing time in these restricted settings.

One of the special cases that Bhatnagar et al. considered was the so-called *k-attribute model*. In this setting, each man and woman has two k -dimensional vectors associated with them, a “preference” vector and a “position” (or “at-

tribute”) vector. A man M_i has a preference vector denoted by \hat{M}_i , and a position vector denoted by \bar{M}_i . Similarly, a woman w_h has a preference vector \hat{w}_j and a position vector \bar{w}_j . Then, M_i prefers w_j over w_k (i.e. w_j appears higher on his preference list than w_k) if and only if $\hat{M}_i \cdot \bar{w}_j > \hat{M}_i \cdot \bar{w}_k$, where $\hat{M}_i \cdot \bar{w}_j$ denotes the usual k -dimensional dot product of vectors.¹ Since we assume that each man has a total order over the women (and vice-versa), a valid instance has the property that $\hat{M}_i \cdot \bar{w}_j \neq \hat{M}_i \cdot \bar{w}_k$ whenever $j \neq k$ (and analogously for the women’s preference vectors/men’s position vectors). In this paper we consider the k -attribute model for the roommate problem.

We also study the stable roommate problem in the k -Euclidean model which we had introduced in a previous paper [3]. In the k -Euclidean model, each person has two associated points in k -dimensional Euclidian space — a “preference” point and a “position” point. The preference point of a person X is denoted by \hat{X} , and the position point is denoted by \bar{X} . Then, X prefers y over z (i.e. y appears higher on his/her preference list than z) if and only if $|\hat{X} - \bar{y}| < |\hat{X} - \bar{z}|$, where $|\hat{X} - \bar{y}|$ denotes the usual k -dimensional Euclidean distance. Once again, a valid instance has the property that $|\hat{X} - \bar{y}| \neq |\hat{X} - \bar{z}|$ whenever $y \neq z$.

We examined the stable marriage problem in our previous paper [3], providing complexity-theoretic evidence for the difficulty of approximately counting stable matchings in both the k -attribute model and the k -Euclidean model. We constructed *approximation-preserving reductions* between (i) counting the number of stable matchings in the k -attribute marriage problem ($k \geq 3$) and counting independent sets in a bipartite graph ($\#BIS$), and (ii) counting the number of stable marriages in the k -Euclidean marriage problem ($k \geq 2$) and $\#BIS$.

Informally speaking, if there is an approximation-preserving reduction (AP-reduction) from one problem to another, then an FPRAS for the second problem implies the existence of an FPRAS for the first. We write $f \leq_{AP} g$ to mean that f has an AP-reduction to g . Similarly, we write $f \equiv_{AP} g$ to mean that $f \leq_{AP} g$ and $g \leq_{AP} f$, or that f and g are AP-interreducible. Approximation-preserving reductions play a role in approximate counting analogous to the role that polynomial many-one reductions play in the theory of NP-completeness and polynomial Turing reductions play in the theory of $\#P$ -completeness.

The complexity class $\#R\P_{11}$ of counting problems was introduced by Dyer, Goldberg, Greenhill and Jerrum [5] as a means to classify approximate counting problems. The problems in $\#R\P_{11}$ are those that can be expressed in terms of counting the number of models of a logical formula from a certain syntactically restricted class which is also known as “restricted Krom SNP” [4]. The complexity

¹ Recall that the dot product of two vectors $\mathbf{a} = (a_1, \dots, a_k)$ and $\mathbf{b} = (b_1, \dots, b_k)$ is the sum $\sum_{i=1}^k a_i b_i$ which is equal to $\|\mathbf{a}\| \|\mathbf{b}\| \cos \theta$, where $\|\mathbf{x}\|$ denotes the length of a vector x and θ is the angle between \mathbf{a} and \mathbf{b} .

class $\#R\Pi_1$ has a completeness class (with respect to AP-reductions) which includes many natural counting problems including: $\#BIS$, counting downsets in a partial order, counting configurations in the Widom-Rowlinson model (all [5]) and computing the partition function of the ferromagnetic Ising model with a mixed external field [7]. Either all of these problems have an FPRAS, or none do. No FPRAS is currently known for any of them, despite much effort having been expended on finding one. More background and details about AP-reducibility are given in Section 2.

Before we continue, we define the problems that are of interest to us in this paper.

Name. $\#SR$

Instance. A stable roommate instance with $2n$ people.

Output. The number of stable roommate assignments.

Name. $\#k$ -attribute SR

Instance. A stable roommate instance with $2n$ people, i.e. preference lists are determined using dot products between k -dimensional preference and position vectors as described above.

Output. The number of stable roommate assignments.

Name. $\#k$ -Euclidean SR.

Instance. A stable roommate instance with $2n$ people. In this setting, each person has a “preference point” and “position point”. Preference lists are determined using Euclidean distances between preference points and position points as described above.

Output. The number of stable roommate assignments.

We also define two other counting problems which are relevant to our results.

Name. $\#IS$.

Instance. A graph G .

Output. The number of independent sets (of all sizes) of G .

Name. #SAT.

Instance. A boolean formula in conjunctive normal form.

Output. The number of satisfying assignments.

1.1 Our results

Zuckerman [18] has shown that #SAT cannot have an FPRAS unless NP=RP. The same is true of any problem in #P to which #SAT is AP-reducible [5]. For example, it is true of #IS, which is AP-interreducible with #SAT [5]. We have the following results.

Theorem 1. $\#IS \equiv_{AP} \#k\text{-ATTRIBUTE SR}$ for $k \geq 4$.

Theorem 2. $\#IS \equiv_{AP} \#k\text{-EUCLIDEAN SR}$ for $k \geq 3$.

Corollary 3. *For any $k \geq 4$, $\#k\text{-ATTRIBUTE SR}$ is complete for #P with respect to AP-reductions. For any $k \geq 3$, $\#k\text{-EUCLIDEAN SR}$ is complete for #P with respect to AP-reductions. Also, #SR is complete for #P with respect to AP-reductions. None of these problems has an FPRAS unless NP=RP.*

Theorem 4. *For every #1-ATTRIBUTE SR instance I , there are either 1 or 2 stable assignments. Thus, #1-ATTRIBUTE SR can be solved exactly in polynomial time.*

We also show the following results.²

Theorem 5. $\#BIS \leq_{AP} \#3\text{-ATTRIBUTE SR}$.

Theorem 6. $\#BIS \leq_{AP} \#2\text{-EUCLIDEAN SR}$.

The last two results are significant since #BIS is complete for #RHH₁ with respect to approximation-preserving reductions.

² The proofs of Theorems 5 and 6 (in Section 7) borrow constructions from the AP-reductions that we presented in [3] from #BIS to the problem of counting stable matchings in the 3-attribute model and the 2-Euclidean model. However, Theorems 5 and 6 do not follow directly from the results of [3] since, in the stable roommate problem, all people need to rank *all* other people (rather than just ranking people of the opposite sex) and this needs to be incorporated into the geometric constructions.

2 Randomized Approximation Schemes and Approximation-preserving reductions

In this section, we give standard definitions of randomized approximation schemes and AP-reductions. A reader who is already familiar with these concepts may safely skip this section.

A *randomized approximation scheme* is an algorithm for approximately computing the value of a function $f : \Sigma^* \rightarrow \mathbb{R}$. The approximation scheme has a parameter $\varepsilon > 0$ which specifies the error tolerance. A *randomized approximation scheme* for f is a randomized algorithm that takes as input an instance $x \in \Sigma^*$ (e.g., for the problem $\#SR$, the input would be an encoding of a stable roommate instance) and a rational error tolerance $\varepsilon > 0$, and outputs a rational number z (a random variable of the “coin tosses” made by the algorithm) such that, for every instance x ,

$$\Pr [e^{-\varepsilon} f(x) \leq z \leq e^{\varepsilon} f(x)] \geq \frac{3}{4}. \quad (1)$$

The randomized approximation scheme is said to be a *fully polynomial randomized approximation scheme*, or *FPRAS*, if it runs in time bounded by a polynomial in $|x|$ and ε^{-1} .

We now define the notion of an approximation-preserving (AP) reduction. Suppose that f and g are functions from Σ^* to \mathbb{R} . As mentioned before, an AP-reduction from f to g gives a way to turn an FPRAS for g into an FPRAS for f . Here is the formal definition. An *approximation-preserving reduction* from f to g is a randomized algorithm \mathcal{A} for computing f using an oracle for g . The algorithm \mathcal{A} takes as input a pair $(x, \varepsilon) \in \Sigma^* \times (0, 1)$, and satisfies the following three conditions: (i) every oracle call made by \mathcal{A} is of the form (w, δ) , where $w \in \Sigma^*$ is an instance of g , and $0 < \delta < 1$ is an error bound satisfying $\delta^{-1} \leq \text{poly}(|x|, \varepsilon^{-1})$; (ii) the algorithm \mathcal{A} meets the specification for being a randomized approximation scheme for f (as described above) whenever the oracle meets the specification for being a randomized approximation scheme for g ; and (iii) the run-time of \mathcal{A} is polynomial in $|x|$ and ε^{-1} .

According to the definition, approximation-preserving reductions may use randomization and may make multiple oracle calls. Nevertheless, the reductions that we present in this paper are deterministic. Each reduction makes a single oracle call (with $\delta = \varepsilon$) and returns the result of that oracle call. A word of warning about terminology: Subsequent to [5], the notation \leq_{AP} has been used to denote a different type of approximation-preserving reduction which applies to optimization problems. We will not study optimization problems in this paper, so hopefully this will not cause confusion.

3 Background and definitions

We first review some of the relevant background and definitions related to stable matchings. The combinatorial structure present in these problems plays a large role in what follows. Many of the definitions are taken from [11] and [8]. The reader is also referred to Gusfield and Irving’s book [10].

It will also help to have an illustrative example, and for these purposes we give such an example in the Appendix.

3.1 Stable matchings and the rotation poset

Irving’s method for finding a stable matching for a SR instance (or concluding that one doesn’t exist) is a two-phase algorithm [11]. During both phases of the algorithm, the preference lists are shortened in a well-defined manner. If we reach a stage where each person has a single element on his/her list, then pairing these people will create a stable matching. Alternatively, if at any point a person’s preference list becomes empty, we conclude that the instance has no stable matching.

Phase 1 is much akin to the usual Gale-Shapley algorithm for the marriage problem, in that people “propose” to one another, holding the best proposal from the ones received so far. For every person e_i , let h_i denote the person who is (currently) first on e_i ’s list. Following [8], we will say that e_i is *semi-engaged* to h_i if and only if e_i is the bottom entry of h_i ’s list. Note that this is not a symmetric relation — h_i is not necessarily semi-engaged in this case. A person who is not semi-engaged is called *free*.

Phase 1 of Irving’s SR algorithm consists of the following steps:

1. If there is an empty list, then stop, there is no stable assignment.
2. Otherwise, if everyone is semi-engaged, go to Phase 2 (described below).
3. Otherwise, pick an arbitrary free person e_i and do the following: For each person p who is ranked below e_i on h_i ’s list, remove p from h_i ’s list, and remove h_i from p ’s list.

So as Phase 1 proceeds, people’s preference lists shrink. At any point during this phase (or the next), we refer to the shortened preference lists as *short lists*, and we refer to the set of short lists as a *table*. It is proved in [11] that if a short list is empty at the end of Phase 1, then the instance has no stable matching. An example of Phase 1 is in the appendix.

Assuming Phase 1 ends with no empty short list, we proceed to Phase 2. To describe this phase, we need more notation and definitions. For a person e_i , we

are already using h_i to denote the person at the head of her short list, and we use s_i to denote the person who is second on her short list.

Definition 1. *Given a set of short lists, a rotation R is an ordered set of people $E = \{e_1, e_2, \dots, e_k\}$ such that $s_i = h_{i+1}$ for all $i \in \{1, \dots, k-1\}$ and $s_k = h_1$. We will also say that R is exposed in the short lists.*

Note that rotations are defined relative to a given set of short lists. An example is in the appendix.

For a rotation R , we will sometimes write $R = (E, H, S)$, where H is the set of head entries of E , ordered in correspondence with E , and, S is the set of second entries of E , again ordered in correspondence with E .

Definition 2. *Given a rotation $R = (E, H, S)$ for a set of short lists, the elimination of R consists of performing the following operation: for every $s_i \in S$, remove every entry below e_i in s_i 's short list, i.e. move the bottom of s_i 's shortlist up to e_i . Then remove s_i from p 's list for each person p that was just removed from s_i 's list.*

Therefore, the elimination of a rotation results in a new set of short lists, where at least two people's lists have shrunk in length.

Phase 2 of Irving's SR algorithm consists of the following steps:

1. If a short list is empty, then stop, the instance has no stable matching.
2. Otherwise, if each person has exactly one entry on his or her short list, then pairing each person with their head entry is a stable matching.
3. Otherwise, find and eliminate some rotation.

For an example of one round of Phase 2, see the appendix. We note the following property of the short lists, which is easily established from the Phase 1 and Phase 2 procedures.

Property 7. *At the end of Phase 1, and at the end of each round of Phase 2, person A has person B on his/her list if and only if person B has person A on his/her list.*

A SR instance may have many stable matchings. Each such stable matching can be found as a result of *some* sequence of rotation eliminations [8].

The set of rotations exhibits a rich combinatorial structure which has been explored previously by other authors. We review this structure here. To do so, we need still more notation and definitions.

Definition 3. Suppose that $R = (E, H, S)$ is a rotation for a SR instance, i.e. R is exposed in some set of short lists. Define R^d to be the triple (S, E, E^r) , where S and E have the same order as they do in R , and E^r is the backwards cyclic rotation of E . That is, if $E = \{e_1, e_2, \dots, e_k\}$ then $E^r = \{e_2, \dots, e_k, e_1\}$.

R^d has the form of a rotation. If R^d is actually a rotation (i.e. R^d is exposed in the set of short lists during some possible execution of the matching algorithm), then we call R and R^d a dual pair of rotations. Any rotation without a dual is called a singleton rotation.

An ordering relation can be defined on the set of all rotations (singletons and dual pairs).

Definition 4. A rotation R' explicitly precedes a rotation R if there is a person p who satisfies both of the following.

- R contains a triple (e_i, h_i, s_i) with $h_i \neq p$ such that p is above s_i in e_i 's (original) preference list.
- R' removes p from e_i 's list by moving the end of p 's list above e_i .

Definition 5. Π^* is the reflexive transitive closure of the “explicitly precedes” relation. We will use the term “precedes” to refer to this relation. $\Pi^*(R, R')$ means that rotation R precedes R' in this partial order.

Let $Rot(I)$ denote the set of all rotations (singletons and dual pairs) that are exposed during some execution of the two-phase algorithm for a given stable roommate instance I . Then Π^* defines a partial order on $Rot(I)$. We refer to this partial order as the *rotation poset*. As usual when dealing with partial orders, a subset $U \subseteq Rot(I)$ is called a *downset* if $R' \in U$ and $\Pi^*(R, R')$ imply that $R \in U$.

The combinatorial significance of the rotation poset is captured in the following theorem.

Theorem 8. [8, Thm 5.1] *There is a one-to-one correspondence between stable matchings and the downsets in Π^* that contain every singleton rotation and exactly one of each dual pair.*

The rotation poset Π^* has even more structure to it.

Lemma 9. [8, Lemma 5.5] *Let $\{R_1, R_1^d\}$ and $\{R_2, R_2^d\}$ be two dual pairs of rotations and R a singleton rotation. Then*

1. *Neither R_1 nor R_1^d precedes R in Π^* , i.e. only a singleton rotation can precede a singleton.*
2. *$\Pi^*(R_1, R_2)$ if and only if $\Pi^*(R_2^d, R_1^d)$.*

The rotation poset plays a key role in our approximation-preserving reductions. For our reductions, we must define a roommate instance I , then identify $\text{Rot}(I)$, find the precedence relations amongst them (i.e. find Π^*), and show that it agrees with our initial starting problem.

Given an instance I of SR with $2n$ people, Gusfield describes a polynomial-time (in n) algorithm for finding the set of all rotations of I , and for constructing a directed graph D that captures the partial order Π^* [8]. (Note: The transitive reduction of D is isomorphic to the Hasse diagram of Π^* , but D might contain more edges than the covering relations defined by the “explicitly precedes” relation. Still, D has no more than $O(n^2)$ edges.)

Before we demonstrate our constructions, we note one more combinatorial construction that serves to encode the set of stable matchings for a given instance.

3.2 Stable matchings and independent sets

Gusfield defines an additional way to represent the set of stable roommate assignments [8, Section 5.3.2].

Let I denote an instance of SR. Define an undirected graph $G(I)$ as follows: Each nonsingleton rotation of I corresponds to a vertex of $G(I)$. Two rotations R_1 and R_2 are connected by an edge in $G(I)$ if and only if there exists a rotation R (possibly R_1 or R_2 themselves) such that $\Pi^*(R, R_1)$ and $\Pi^*(R^d, R_2)$. In particular, we note that R_1 and R_1^d are connected by an edge for each dual pair $\{R_1, R_1^d\}$ (as a vertex precedes itself, by definition, in the partial order Π^*). See the appendix for an example.

Gusfield also defines another partial order involving only the non-singleton rotations.

Definition 6. *Suppose Σ is the set of all singletons in Π^* . Then $\Pi = \Pi^* - \Sigma$ is a partial order on the set of all non-singleton (dual) rotations.*

Having defined this undirected graph and the partial order Π , we have these results, a combination of Lemmas 5.6 and 5.10, and Theorem 5.3 in [8].

Theorem 10. *Let I denote an instance of SR and $G(I)$ its corresponding graph constructed as above.*

1. *Every maximal independent set in $G(I)$ contains exactly one node from each dual pair of rotations.*
2. *There is a one-to-one correspondence between maximal independent sets in $G(I)$ and stable matchings of I .*
3. *Rotations R_1 and R_2 are connected by an edge in $G(I)$ if and only if R_1^d precedes R_2 , i.e. R_1 and R_2 are connected if and only if $\Pi(R_1^d, R_2)$.*

4 A construction for showing $\#IS \equiv_{AP} \#4\text{-attribute SR}$

Recall that $\#k\text{-ATTRIBUTE SR}$ denotes the problem of counting stable assignments for k -attribute stable roommate instances. Our goal of this section is to prove Theorem 1 which we restate below.

Theorem 1. $\#IS \equiv_{AP} \#k\text{-ATTRIBUTE SR}$ for $k \geq 4$.

Proof. First, since $\#IS$ is complete for $\#P$ with respect to AP-reductions [5], and $\#k\text{-ATTRIBUTE SR} \in \#P$, we immediately have $\#k\text{-ATTRIBUTE SR} \leq_{AP} \#IS$. Also, it is easy to see, for $k > 4$, that $\#4\text{-ATTRIBUTE SR} \leq_{AP} \#k\text{-ATTRIBUTE SR}$ (the reduction uses up the extra $k - 4$ dimensions by assigning some particular value in every preference vector and attribute vector). Thus, it remains to prove $\#IS \leq_{AP} \#4\text{-ATTRIBUTE SR}$. This is proved in the rest of Section 4, including Subsections 4.1 — 4.6. \square

We wish to prove $\#IS \leq_{AP} \#4\text{-ATTRIBUTE SR}$. To this end, let $\Gamma = (V, E)$ be an instance of $\#IS$. Let \mathcal{T} be the Cartesian product $\mathcal{T} = V \times \{0\}$ and let $\mathcal{B} = V \times \{1\}$. The sets \mathcal{T} and \mathcal{B} are just two distinct copies of V . Let E' be the matching on $\mathcal{B} \cup \mathcal{T}$ defined by $E' = \{((v, 0), (v, 1)) \mid v \in V\}$. Let $E'' = \{((v, 0), (w, 0)) \mid (v, w) \in E\}$. The set E'' just mimics the edges of E amongst the vertices of \mathcal{T} . Finally, let $\Gamma' = (\mathcal{B} \cup \mathcal{T}, E' \cup E'')$. Note the bijection between independent sets of Γ , (which we wish to approximately count, using as an oracle a FPRAS for $\#4\text{-ATTRIBUTE SR}$), and maximal independent sets of Γ' . Our goal is to construct a 4-attribute stable roommate instance I so that the graph $G(I)$ is isomorphic to Γ' . This will complete the proof, by Theorem 10. So from now on, we will focus exclusively on constructing I so that $G(I)$ is isomorphic to Γ' . As soon as we've done that, we are finished.

We recall the construction of $G(I)$ from I : The vertices of $G(I)$ are the nonsingleton rotations of I . From Theorem 10, two rotations R_1 and R_2 are connected by an edge of G if and only if $\Pi(R_1^d, R_2)$ (and hence, by Lemma 9, $\Pi(R_2^d, R_1)$).

Our method, given Γ' , will be to construct I so that its nonsingleton rotations can be labelled bijectively with $\mathcal{B} \cup \mathcal{T}$ in such a way that the following conditions are satisfied (by the partial order Π associated with I).

For all $v \in V$, the rotation labelled $(v, 0)$ is dual to the one labelled $(v, 1)$. (2)

Also, using the vertices of Γ' to refer to the corresponding rotations of I ,

$$\{(R, R') \in \mathcal{T} \times \mathcal{T} \mid \Pi(R^d, R')\} = E'', \quad (3)$$

$$\{(R, R') \in \mathcal{T} \times \mathcal{T} \mid \Pi(R, R'^d)\} = \emptyset, \quad (4)$$

$$\{(R, R') \in \mathcal{T} \times \mathcal{T} \mid \Pi(R, R')\} = \{(R, R') \in \mathcal{T} \times \mathcal{T} \mid R = R'\}. \quad (5)$$

Under the assumption that (2) holds,

- Equation (3) guarantees that $E(G(I)) \cap (\mathcal{T} \times \mathcal{T}) = E''$.
- Equation (4) guarantees that $E(G(I)) \cap (\mathcal{B} \times \mathcal{B}) = \emptyset$.
- Finally, equation (5) guarantees that $E(G(I)) \cap (\mathcal{B} \times \mathcal{T}) = E'$.

Thus, Equations (2)–(5), taken together, guarantee that $G(I)$ is isomorphic to Γ' , as required. So all that we need to do, to complete the reduction, and the proof, is to use the input graph Γ to construct an instance I so that its nonsingleton rotations can be labelled bijectively with $\mathcal{B} \cup \mathcal{T}$ in such a way that Equations (2)–(5) are satisfied. We concentrate on this for the rest of the proof. (We never need to consider Γ' again.)

Unfortunately, the notation that we used to get this far (which came from earlier papers) is not going to be very convenient when we come to actually do the construction. So, in order to make the following (rather complicated!) proof easier to follow, we are now going to change notation.

First, instead of using the graph Γ as the input to our construction, we will instead take as input a bipartite graph K which captures all of the information about $\Gamma = (V, E)$. The bipartite graph K will have vertex partition $B = \{b_1, \dots, b_n\}$, $T = \{t_1, \dots, t_n\}$ and edge set $E(K)$ satisfying the following two properties,

$$(K1) \quad (b_i, t_i) \notin E(K) \quad \forall i \in [n],$$

$$(K2) \quad (b_i, t_j) \in E(K) \text{ if and only if } (b_j, t_i) \in E(K).$$

The correspondence between K and our original graph Γ is as follows: We take n to be $|V|$ so that there is a natural bijection between B and the set $\mathcal{B} = V \times \{1\}$ defined above. Also, there is a natural bijection between T and $\mathcal{T} = V \times \{0\}$. The edge set $E(K)$ is constructed from the edge set E of Γ as follows. Suppose that $(u, 1)$ is the i 'th element of \mathcal{B} and that $(v, 0)$ is the j 'th element of \mathcal{T} . The edges (b_i, t_j) and (b_j, t_i) are included in $E(K)$ if and only if (u, v) is an edge of E . The reader should verify that the graph K encodes all of the information about the input graph Γ , in the sense that we could reconstruct Γ given K . Also, for every undirected graph Γ , there is a corresponding K , and it can be constructed in polynomial time.

The sole problem remaining (and it is a big one!) is to show how, given K , (and therefore deducing Γ and Γ'), to construct a $\#4$ -ATTRIBUTE SR instance I so that its nonsingleton rotations can be labelled bijectively with $\mathcal{B} \cup \mathcal{T}$ in such a way that Equations (2)–(5) are satisfied. As soon as we accomplish that, we have finished the reduction and the proof.

The whole point of introducing the bipartite graph K is that we can re-state the Equations (2)–(5) in a manner that will be more convenient to work with. In particular, using the bijection between B and \mathcal{B} and the bijection between T and \mathcal{T} , these equations are equivalent to the following.

- (G1) For all $i \in [n]$, the rotation labeled b_i is dual to the rotation labeled t_i .
- (G2) The set of pairs (b_i, t_j) in Π is $E(K)$.
- (G3) There are no pairs (t_i, b_j) in Π .
- (G4) There are no pairs (t_i, t_j) in Π except for those with $i = j$ (which are all present).

4.1 The task remaining

We have finally defined all of the conditions that we need. At this point, the reader should have verified that the sole problem remaining is, given a bipartite graph K satisfying (K1)–(K2), we must show how to construct, in polynomial time, a #4-ATTRIBUTE SR-instance I satisfying (G1)–(G4). Once we do that, we are finished with the reduction and the proof. We will not have to further consider the graphs Γ and Γ' . These were needed only to get to this point.

4.2 The construction of I

Our first task will be to show how to construct I , given K , and for this it will be helpful to define some notation for describing the bipartite graph K . Similar to the construction defined in [3], label the edges in $E(K)$ according to the lexicographic order of the pair (b_i, t_j) (so the edges with the smallest labels are incident to b_1). The first edge incident to b_1 is given the label $(1, 2)$, the second is given the label $(3, 4)$ and so on. The i th edge in the lexicographic ordering is given the label $(2i - 1, 2i)$. Let m denote the number of edges in K . The roommate instance we construct will have $4m$ people in it, which we denote as $P_1, \dots, P_{2m}, Q_1, \dots, Q_{2m}$.

We define two permutations ρ and σ of $[2m]$ as in [3], so the first ρ -cycle corresponds to the labels of the edges incident on b_1 , the first σ -cycle corresponds to the labels of the edges incident on t_1 , and so on. We explain this with the help of the example in Figure 1.

From Figure 1, it is clear that the i th ρ -cycle is obtained by taking together the labels of the edges incident at node b_i . The first σ -cycle involves the labels of edges incident to t_1 . Vertex t_1 has only one edge incident to it, namely edge $(3, 4)$. Hence, the first σ -cycle is $(3, 4)$. The second σ -cycle involves edges incident to t_2 , namely, edges $(1, 2), (9, 10)$ and $(13, 14)$. The second σ -cycle is obtained by grouping

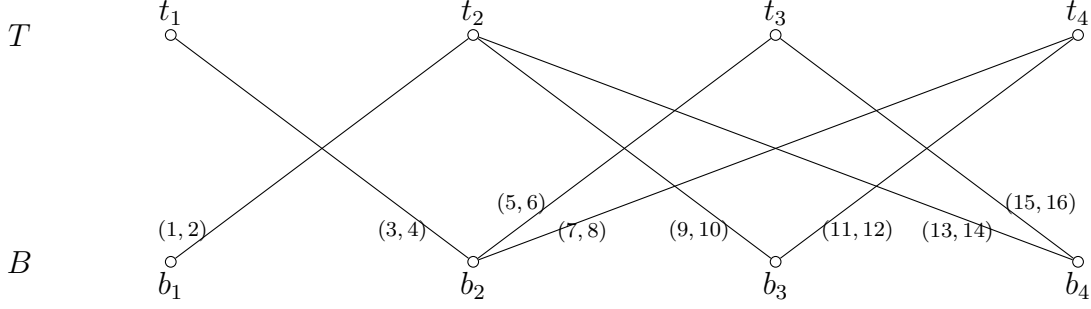


Figure 1: Defining ρ -cycles and σ -cycles from K .
 ρ -cycles : $(1,2), (3,4,5,6,7,8), (9,10,11,12), (13,14,15,16)$
 σ -cycles : $(3,4), (1,2,9,10,13,14), (5,6,15,16), (7,8,11,12)$

together the labels of the three edges and the σ -cycle is $(1, 2, 9, 10, 13, 14)$. In this manner, we obtain the remaining σ -cycles.

Note that here we have more structure than was present in the construction from [3] — (K1) and (K2) ensure that the number of ρ -cycles is equal to the number of σ -cycles, and the number of elements in the i th ρ -cycle is identical to the number of elements in the i th σ -cycle.

Suppose there are n ρ -cycles, and, hence, n σ -cycles. Suppose also that $|\rho_i| = q_i$, i.e. the i th ρ -cycle consists of q_i elements. Thus, the i th σ -cycle also has q_i elements and q_i is even. Note that for $1 \leq k \leq q_i/2$, the elements $2k - 1$ and $2k$ are in the same ρ -cycle and they are in the same σ -cycle. Also, a given ρ -cycle and a given σ -cycle intersect in at most one such pair. In what follows, we let the “representative” of each ρ cycle be the (numerically) smallest number in the cycle. In Figure 1, the representatives of the ρ cycles are elements 1, 3, 9 and 13. Note that each representative of a ρ -cycle is an odd number. We let $Rep(\rho) = \{f_1, \dots, f_n\}$ denote the set of representatives, so, for each $i \in \{1, \dots, n\}$, the cycle ρ_i can be represented as $\rho_i = (f_i, \rho f_i, \rho^2 f_i, \dots, \rho^{q_i-1} f_i)$.

Let $\psi : [m] \rightarrow [m]$ be the bijection defined so that, if ℓ is the k th element of the i th ρ -cycle, then $\psi(\ell)$ is the k th element of the i th σ -cycle. So if the i ’th ρ -cycle is (i_1, \dots, i_d) then the i ’th σ -cycle is $(\psi(i_1), \dots, \psi(i_d))$. From the properties of the bipartite graph K , we note that ψ is an involution, i.e. $\psi \circ \psi = \text{identity}$, where $\psi \circ \psi$ is usual function composition. Let $Rep(\sigma) = \{\psi(f_i) \mid f_i \in Rep(\rho)\}$ be the set of representatives of the n σ -cycles.

Recall that to specify our roommate instance I , we must define 4-dimensional position vectors $\overline{P}_i, \overline{Q}_i$ and preference vectors \hat{P}_i, \hat{Q}_i for each person.

Before doing this, we give a quick look ahead at what is to come in the construction. First of all, the instance we specify will have no singleton rotations. In

order to satisfy condition (G1), the pair (b_i, t_i) will correspond to a dual pair of rotations. The rotation associated with the vertex b_i will relate to the i th ρ -cycle. Suppose that this cycle is (i_1, \dots, i_d) . Then the rotation will be

$$\begin{array}{c|c} Q_{i_1} & P_{i_1}P_{i_2} \\ Q_{i_2} & P_{i_2}P_{i_3} \\ \dots & \dots \\ Q_{i_d} & P_{i_d}P_{i_1}. \end{array} \quad (6)$$

Recalling Definition 3, the dual rotation, associated with the vertex t_i , is

$$\begin{array}{c|c} P_{i_1} & Q_{i_d}Q_{i_1} \\ P_{i_2} & Q_{i_1}Q_{i_2} \\ \dots & \dots \\ P_{i_d} & Q_{i_{d-1}}Q_{i_d}. \end{array} \quad (7)$$

In order to satisfy condition (G2) we need, for $(b_i, t_j) \in E(K)$, for rotation b_i to precede rotation t_j . These dependencies will be captured in our construction by making sure that other (appropriately chosen) P people are on the preference lists of the P people, between the Q s given on the lists in (7).

The remaining parts of this section are devoted to giving the detailed construction of our 4-dimensional roommate instance I (which will have no singleton rotations) and showing that its nonsingleton rotations are exactly those given in (6) and (7). Using this bijection between the rotations and $B \cup T$ we then show that (G1)–(G4) are satisfied, as required in Section 4.1.

4.3 Assigning position and preference vectors

We start by assigning the first two coordinates of the position vectors. The people Q_1, \dots, Q_{2m} have 0 in each of these coordinates. The first two coordinates of the positions of P_1, \dots, P_{2m} are arranged around a unit circle, taking each ρ -cycle in order (and leaving a big gap before the next ρ -cycle). Let $\epsilon = \frac{2\pi}{(2m)^2}$. The i 'th ρ cycle takes up an angle of ϵ (out of the 2π radians around the circle), starting at an angle of $2\pi(i-1)/n$. Let $\theta_i = \epsilon/(7(q_i-1))$. If the i 'th ρ -cycle is the cycle (i_1, \dots, i_d) , then positions $\overline{P}_{i_1}, \dots, \overline{P}_{i_d}$ are assigned in order, leaving a gap of $7\theta_i$ between each pair of people.

More formally, we define the first two coordinates of the position vectors as follows (the asterisks in the second two coordinates will be defined shortly):

For $f_i \in \text{Rep}(\rho)$, for $0 \leq k \leq q_i - 1$, set

$$\begin{aligned}\overline{P}_{\rho^k f_i} &= (\cos(2\pi(i-1)/n + 7k\theta_i), \sin(2\pi(i-1)/n + 7k\theta_i), *, *) \quad \text{and} \\ \overline{Q}_{\rho^k f_i} &= (0, 0, *, *).\end{aligned}$$

We next assign the last two coordinates of the positions. Once again, these coordinates are arranged around a unit circle, taking each ρ -cycle in order and leaving big gaps between consecutive ρ -cycles. The i 'th ρ cycle takes up an angle of up to ϵ starting at an angle of $2\pi(i-1)/n$. Let $\theta'_i = \epsilon/4$. If the i 'th ρ -cycle is (i_1, \dots, i_d) , then positions are assigned in the following order:

$$\overline{Q}_{i_1} \overline{P}_{\psi(i_2)} \overline{Q}_{i_2} \overline{P}_{\psi(i_3)} \overline{Q}_{i_3} \overline{P}_{\psi(i_4)} \cdots \overline{Q}_{i_{d-1}} \overline{P}_{\psi(i_d)} \overline{Q}_{i_d} \overline{P}_{\psi(i_1)}.$$

For $k \in \{1, \dots, d-1\}$, the angle between \overline{Q}_{i_k} and $\overline{Q}_{i_{k+1}}$ is $2^{-(k-1)}2\theta'_i$. Also, $\overline{P}_{\psi(i_{k+1})}$ is at an equal angle between these.

To simplify the notation while assigning the i 'th ρ -cycle (i_1, \dots, i_d) , let i_{d+1} denote i_1 . The second coordinates of the position vectors are defined in the following manner. Note that the asterisks representing values in the first two coordinates have already been defined above.

For $f_i \in \text{Rep}(\rho)$, for $0 \leq k \leq q_i - 1$, set

$$\begin{aligned}\overline{Q}_{\rho^k f_i} &= \left(*, *, \cos\left(\frac{2\pi(i-1)}{n} + 2\theta'_i \sum_{j=0}^{k-1} 2^{-j}\right), \sin\left(\frac{2\pi(i-1)}{n} + 2\theta'_i \sum_{j=0}^{k-1} 2^{-j}\right) \right) \quad \text{and} \\ \overline{P}_{\psi(\rho^{k+1} f_i)} &= \left(*, *, \cos\left(\frac{2\pi(i-1)}{n} + 2\theta'_i \sum_{j=0}^{k-1} 2^{-j} + 2^{-k}\theta'_i\right), \right. \\ &\quad \left. \sin\left(\frac{2\pi(i-1)}{n} + 2\theta'_i \sum_{j=0}^{k-1} 2^{-j} + 2^{-k}\theta'_i\right) \right).\end{aligned}$$

A sum of the form $\sum_{j=0}^{-1} 2^{-j}$ is taken to be equal to 0.

Having defined the position vectors, we now give the preference vectors. These are also defined using the ρ -cycles, and again are placed around a unit circle.

The preference vectors \widehat{Q}_j are 0 in the last two coordinates. If the i 'th ρ -cycle is (i_1, \dots, i_d) , then, in the first two coordinates, for $1 \leq j < d$, \widehat{Q}_{i_j} is placed between \overline{P}_{i_j} and $\overline{P}_{i_{j+1}}$, slightly closer to \overline{P}_{i_j} . Here is the definition. For $f_i \in \text{Rep}(\rho)$, for $0 \leq k \leq q_i - 1$, set

$$\widehat{Q}_{\rho^k f_i} = (\cos(2\pi(i-1)/n + 7k\theta_i + 3\theta_i), \sin(2\pi(i-1)/n + 7k\theta_i + 3\theta_i), 0, 0).$$

Suppose that the i 'th ρ -cycle is (i_1, \dots, i_d) . Looking at the preference vectors \widehat{Q}_{i_j} and the position vectors \overline{P}_{i_j} , we conclude that, for $j < d$, the preference list of Q_{i_j} will start $P_{i_j}P_{i_{j+1}}$. This is consistent with our desired rotation (6). The preference list of Q_{i_d} will start $P_{i_d}\{P_{i_{d-1}} \dots P_{i_2}\}P_{i_1}$. We refer to the part contained in $\{\}$ symbols as “noise” on the preference list. We will have to show that this “noise” does not introduce any extra rotations aside from the ones we desire.

The preference vectors \widehat{P}_j are 0 in the first two coordinates. If the i 'th ρ -cycle is (i_1, \dots, i_d) , then, in the last two coordinates, for $j > 1$, \widehat{P}_{i_j} is placed $1/3$ of the way along between $\overline{Q}_{i_{j-1}}$ and $\overline{P}_{\psi(i_j)}$. Then, \widehat{P}_{i_1} is placed between \overline{Q}_{i_d} and $\overline{P}_{\psi(i_1)}$, slightly nearer to \overline{Q}_{i_d} . Here is the definition. For $f_i \in \text{Rep}(\rho)$, for $0 \leq k \leq q_i - 1$, set

$$\widehat{P}_{\rho^{k+1}f_i} = \left(0, 0, \cos\left(\frac{2\pi(i-1)}{n} + 2\theta'_i \sum_{j=0}^{k-1} 2^{-j} + \frac{1}{3}2^{-k}\theta'_i\right), \sin\left(\frac{2\pi(i-1)}{n} + 2\theta'_i \sum_{j=0}^{k-1} 2^{-j} + \frac{1}{3}2^{-k}\theta'_i\right)\right).$$

Suppose that the i 'th ρ -cycle is (i_1, \dots, i_d) . Looking at the preference vectors \widehat{P}_{i_j} and the last two coordinates of the position vectors, we conclude that, for $j > 1$, the preference list of P_{i_j} starts $Q_{i_{j-1}}P_{\psi(i_j)}Q_{i_j}$. This is consistent with the rotation (7) except for the $P_{\psi(i_j)}$ in the second position. This is by design, and will help to ensure the desired precedences between the rotations. The preference list of P_{i_1} starts

$$Q_{i_d}P_{\psi(i_1)}\{P_{\psi(i_d)}Q_{i_{d-1}} \dots P_{\psi(i_3)}Q_{i_2}P_{\psi(i_2)}\}Q_{i_1} \dots,$$

where, once again, the part of the list in $\{\}$ is “noise”, which is not desired, but will turn out to do no harm.

Having given our construction, we have to show that the rotations that get exposed are exactly the ones that we have identified in (6) and (7). Using this bijection between the rotations and $B \cup T$ we will then show that (G1)–(G4) are satisfied, as required in Section 4.1.

We start by summarising the observations that we have made about the preference lists. Consider one of the ρ -cycles, $\rho_i = (i_1, \dots, i_d)$. Using our position and preference vectors, we can write down prefixes of the preference lists for the people $P_{i_1}, \dots, P_{i_d}, Q_{i_1}, \dots, Q_{i_d}$ in this ρ -cycle.

$$\begin{array}{l|l} Q_{i_1} & P_{i_1}P_{i_2} \dots \\ Q_{i_2} & P_{i_2}P_{i_3} \dots \\ \dots & \dots \\ Q_{i_{d-1}} & P_{i_{d-1}}P_{i_d} \dots \\ Q_{i_d} & P_{i_d}\{P_{i_{d-1}} \dots P_{i_2}\}P_{i_1} \dots \end{array} \quad (8)$$

$$\begin{array}{c|c}
P_{i_1} & Q_{i_d} P_{\psi(i_1)} \{P_{\psi(i_d)} Q_{i_{d-1}} \cdots P_{\psi(i_3)} Q_{i_2} P_{\psi(i_2)}\} Q_{i_1} \cdots \\
P_{i_2} & Q_{i_1} P_{\psi(i_2)} Q_{i_2} \cdots \\
\cdots & \cdots \\
P_{i_{d-1}} & Q_{i_{d-2}} P_{\psi(i_{d-1})} Q_{i_{d-1}} \cdots \\
P_{i_d} & Q_{i_{d-1}} P_{\psi(i_d)} Q_{i_d} \cdots
\end{array} \tag{9}$$

We have not given the entire preference lists, but only the parts that are relevant for us. As we show in the next section, the portion of each preference list that is given above is the only part that remains after Phase 1 of Irving's Roommate Algorithm. In fact, additional parts of some preference lists, specifically the parts listed in braces, vanish during the execution of Phase 1.

Example. The prefixes of the preference lists for the example from Figure 1 are given in the second appendix. \square

Remark. Strictly speaking, the construction that we have given does not fit into our original definition of a roommate instance. In particular, with the position and preference vectors we have defined, we note that $\hat{Q}_i \cdot \overline{Q}_j = 0$ for all $i \neq j$. This means that each Q_i has a tie in his/her preference for all of the other Q_j s ($j \neq i$). However, it is easy to modify the position vectors \overline{Q}_j so that we have a strict preference ordering for each person. We can also do this in such a way that the *beginning* of the preference lists of the Q_i s is undisturbed. We may, for example, pick some very small $\delta_j > 0$ and assign the first two coordinates of \overline{Q}_j to equal δ_j , resulting in a strict preference list for Q_i . By choosing the δ_j s small enough, we can get a strict preference for all of the people without altering the beginning segment of the preference list of each person. In particular, as long as the start of the preference lists given in (8) and (9) is maintained, this is sufficient for our purposes. \square

4.4 Enumerating rotations

In this section, we will establish the rotations in the stable roommate instance I . First, consider Phase 1. From (8) and (9) we see that, during Phase 1, each Q_{i_j} will become semi-engaged to P_{i_j} and each P_{i_j} will become semi-engaged to $Q_{i_{j-1}}$. Since the outcome of Phase 1 is independent of the order in which free people make proposals, we can assume that these proposals occur in order. Thus, the suffixes of the preference lists that are omitted from (8) and (9) will disappear by the end of Phase 1. The purpose of this section is to show that, by the end of Phase 1, the preference lists look like (11) and (12).

By construction, the length of each cycle is even. If $d = 2$ then the preference list of Q_{i_d} in (8) has no noise. Otherwise, $P_{i_2}, \dots, P_{i_{d-1}}$ do not have Q_{i_d} on their

short lists in (9) so, since the process of removing people from each other's lists is symmetric, we can conclude that, by the end of Phase 1, these people are removed from the short list of Q_{i_d} .

If $d = 2$ then the preference list of P_{i_1} has prefix $Q_{i_2}P_{\psi(i_1)}P_{\psi(i_2)}Q_{i_1}$. Since i_1 is the representative of the ρ -cycle (i_1, i_2) , it is odd, so i_2 is even and therefore $\psi(i_2)$ is even, and is therefore not the representative of any ρ -cycle. Thus, the preference list of $P_{\psi(i_2)} = P_{\psi(i_d)}$ starts with $Q_{\rho^{-1}(\psi(i_d))}P_{\psi^2(i_d)}Q_{\psi(i_d)} = Q_{\rho^{-1}(\psi(i_d))}P_{i_d}Q_{\psi(i_d)}$. P_{i_1} comes after this prefix on the preference list. Thus, P_{i_1} is removed from this preference list by the end of Phase 1. Symmetrically, $P_{\psi(i_d)}$ is removed from the preference list of P_{i_1} by the end of Phase 1.

Now suppose $d > 2$. Suppose that (i_1, \dots, i_d) is the k 'th ρ -cycle. Now $Q_{i_2}, \dots, Q_{i_{d-1}}$ do not have P_{i_1} on their short lists in (8) so we can conclude that, by the end of Phase 1, these people are removed from the short list of P_{i_1} . For $\ell \in \{2, \dots, d\}$, consider the person $P_{\psi(i_\ell)}$, which is on the short list of P_{i_1} in (9). If $\psi(i_\ell)$ is not the representative of a ρ -cycle, then, using the same argument that we used in the $d = 2$ case, the preference list of $P_{\psi(i_\ell)}$ starts with $Q_{\rho^{-1}(\psi(i_\ell))}P_{i_\ell}Q_{\psi(i_\ell)}$, so P_{i_1} follows this prefix and $P_{\psi(i_\ell)}$ is removed from the preference list of P_{i_1} by the end of Phase 1. Suppose that $\psi(i_\ell)$ is the representative of a ρ -cycle, say the t 'th ρ -cycle. To ease notation, suppose that $\psi(i_\ell) = j_1$ and that $\rho_t = (j_1, \dots, j_d)$. From (9), we know that the preference list of P_{j_1} starts with the following prefix

$$Q_{j_{d'}}P_{\psi(j_1)}\{P_{\psi(j_{d'})}Q_{j_{d'-1}} \cdots P_{\psi(j_3)}Q_{j_2}P_{\psi(j_2)}\}Q_{j_1}. \quad (10)$$

Then $\psi(j_1) = i_\ell$ is a member of the t -th σ -cycle and the k 'th ρ -cycle. $\psi(i_\ell)$ is odd, so i_ℓ is odd, so i_ℓ and $i_\ell + 1 (= i_{\ell+1})$ are the only two elements that are in both of these cycles. We conclude that i_1 is not in both of these cycles, so it is not in the t 'th σ -cycle. Thus, none of the people $P_{\psi(j_{d'})} \cdots P_{\psi(j_3)}P_{\psi(j_2)}$ on the prefix (10), all of whom are in the t 'th σ -cycle, is equal to P_{i_1} . Therefore, P_{i_1} comes after the prefix depicted in (10). As above, we can conclude that $P_{\psi(i_\ell)}$ is removed from the preference list of P_{i_1} by the end of Phase 1.

Hence, after Phase 1 the short lists look like this.

$$\begin{array}{c|c} Q_{i_1} & P_{i_1}P_{i_2} \\ Q_{i_2} & P_{i_2}P_{i_3} \\ \cdots & \cdots \\ Q_{i_{d-1}} & P_{i_{d-1}}P_{i_d} \\ Q_{i_d} & P_{i_d}P_{i_1} \end{array} \quad (11)$$

$$\begin{array}{c|c} P_{i_1} & Q_{i_d}P_{\psi(i_1)}Q_{i_1} \\ P_{i_2} & Q_{i_1}P_{\psi(i_2)}Q_{i_2} \\ \cdots & \cdots \\ P_{i_{d-1}} & Q_{i_{d-2}}P_{\psi(i_{d-1})}Q_{i_{d-1}} \\ P_{i_d} & Q_{i_{d-1}}P_{\psi(i_d)}Q_{i_d} \end{array} \quad (12)$$

Example. The short lists for the example from Figure 1 are given in the second appendix. \square

After Phase 1, we know that for each ρ -cycle, each Q_{i_j} is semi-engaged to P_{i_j} and each P_{i_j} is semi-engaged to $Q_{i_{j-1}}$. According to Section 4.1 we now need to identify the rotations, and to establish conditions (G1)–(G4). Then we are finished.

Lemma 11. *Suppose the n ρ -cycles of the bipartite graph K are $\rho_1, \rho_2, \dots, \rho_n$, where $\rho_j = (i_{j-1} + 1, \dots, i_j)$ for $j \in [n]$, with $i_0 = 0$ and $i_n = 2m$.*

The shortlists obtained after Phase 1 have exactly n exposed rotations, R_1 through R_n , where $R_j = (E_j, H_j, S_j)$,

$$\begin{aligned} E_j &= \{Q_{i_{j-1}+1}, Q_{i_{j-1}+2}, \dots, Q_{i_j}\}, \\ H_j &= \{P_{i_{j-1}+1}, P_{i_{j-1}+2}, \dots, P_{i_j}\}, \text{ and} \\ S_j &= \{P_{i_{j-1}+2}, P_{i_{j-1}+3}, \dots, P_{i_j}, P_{i_{j-1}+1}\}. \end{aligned}$$

Proof. By construction, $\text{Rep}(\rho) = \{i_0 + 1, i_1 + 1, \dots, i_{n-1} + 1\}$. From (11) and (12), the short lists of the Q_* and the P_* people after Phase 1 can be summarized as follows:

$$\begin{aligned} \text{For } j \in \text{Rep}(\sigma), \\ Q_{\rho^k j} &: P_{\rho^k j} P_{\rho^{k+1} j}, \quad 0 \leq k \leq q_j - 1, \\ P_{\rho^k j} &: Q_{\rho^{(k-1)} j} P_{\psi(\rho^k j)} Q_{\rho^k j}, \quad 0 \leq k \leq q_j - 1. \end{aligned}$$

We observe that the Q_* people whose indices belong to ρ_j , along with their first and second preferences, form an exposed rotation $R_j = (E_j, H_j, S_j)$, where E_j, H_j , and S_j are as defined in the statement of the lemma. This is a rotation as displayed in (6). Therefore, at the end of Phase 1, the stable roommate instance I has *at least* n exposed rotations. Further, each Q_* person appears in one of these n rotations, meaning that each Q_* appears in one of the sets E_j .

Next, we show that the above n rotations are the *only* rotations that are exposed after Phase 1. Suppose person P_i is in an exposed rotation $R = (E, H, S)$, i.e. $P_i \in E$. The initial part of the preference list of P_i at the end of Phase 1 is $Q_{\rho^{-1}i} P_{\psi(i)}$. Hence, by definition, the next person in the ordered set E is $Q_{\psi(i)}$ who is currently semi-engaged to $P_{\psi(i)}$. Suppose $\psi(i)$ belongs to the k th ρ -cycle ρ_k . We know that $Q_{\psi(i)}$ is part of the exposed rotation $R_k = (E_k, H_k, S_k)$, i.e. $Q_{\psi(i)} \in E_k = \{Q_j : j \in \rho_k\}$. This implies that $P_i \in E_k$ which is not possible.

Hence, if $R = (E, H, S)$ is an exposed rotation in the table obtained after Phase 1, then $P_i \notin E$ for all i . Therefore, there are exactly n exposed rotations R_1, R_2, \dots, R_n at the end of Phase 1. \square

Example. For the example from Figure 1, there are 4 exposed rotations, $R_1 = \{E_1, H_1, S_1\}$, $R_2 = \{E_2, H_2, S_2\}$, $R_3 = \{E_3, H_3, S_3\}$ and $R_4 = \{E_4, H_4, S_4\}$. These are given as follows.

$$\begin{aligned}
E_1 &= \{Q_1, Q_2\}, \\
H_1 &= \{P_1, P_2\}, \\
S_1 &= \{P_2, P_1\}, \\
E_2 &= \{Q_3, Q_4, Q_5, Q_6, Q_7, Q_8\}, \\
H_2 &= \{P_3, P_4, P_5, P_6, P_7, P_8\}, \\
S_2 &= \{P_4, P_5, P_6, P_7, P_8, P_3\}, \\
E_3 &= \{Q_9, Q_{10}, Q_{11}, Q_{12}\}, \\
H_3 &= \{P_9, P_{10}, P_{11}, P_{12}\}, \\
S_3 &= \{P_{10}, P_{11}, P_{12}, P_9\}, \\
E_4 &= \{Q_{13}, Q_{14}, Q_{15}, Q_{16}\}, \\
H_4 &= \{P_{13}, P_{14}, P_{15}, P_{16}\}, \\
S_4 &= \{P_{14}, P_{15}, P_{16}, P_{13}\}.
\end{aligned}$$

□

Now we show that each of the n rotations from Lemma 11 has a dual rotation of the form (7).

Lemma 12. *For $j \in [n]$, there is a dual rotation $R_j^d = \{E_j^d, H_j^d, S_j^d\}$ corresponding to rotation R_j from Lemma 11, where*

$$\begin{aligned}
E_j^d &= S_j = \{P_{i_{j-1}+2}, P_{i_{j-1}+3}, \dots, P_{i_j}, P_{i_{j-1}+1}\}, \\
H_j^d &= E_j = \{Q_{i_{j-1}+1}, Q_{i_{j-1}+2}, \dots, Q_{i_j}\}, \\
S_j^d &= E_j^r = \{Q_{i_{j-1}+2}, \dots, Q_{i_j}, Q_{i_{j-1}+1}\}.
\end{aligned}$$

Proof. First we note that R_j^d as defined above has the *form* of a rotation and will be the dual to R_j *provided* that there is some sequence of rotations that can be performed that leads to R_j^d being exposed in the resulting table. We show that there is such a sequence of rotations.

Let $\rho_j = (i_{j-1} + 1, i_{j-1} + 2, \dots, i_j)$ denote the ρ -cycle corresponding to the rotation R_j , where $q_j = i_j - i_{j-1} = |\rho_j|$. The j 'th σ -cycle, σ_j , also has $|\sigma_j| = q_j$, so write $\sigma_j = (\ell_1, \dots, \ell_{q_j}) = (\psi(i_{j-1} + 1), \psi(i_{j-1} + 2), \dots, \psi(i_j))$. Recall that $\rho_j \cap \sigma_j = \emptyset$ and that, for any $r \in [n]$, $|\rho_j \cap \sigma_r| \in \{0, 2\}$.

Given σ_j , there are *exactly* $q_j/2$ distinct ρ -cycles $\rho_{t_1}, \dots, \rho_{t_{q_j/2}}$ such that, for $k \in \{1, \dots, q_j/2\}$, $\ell_{2k-1} \in \rho_{t_k}$ and $\ell_{2k} \in \rho_{t_k}$.

Consider the table T obtained at the end of Phase 1. We claim that the elimination of rotations $R_{t_1}, \dots, R_{t_{q_j/2}}$ from T , exposes the (proposed) dual rotation R_j^d . After the elimination of those rotations, every Q_r and P_r , for $r \in \rho_{t_1} \cup \rho_{t_2} \cup \dots \cup \rho_{t_{q_j/2}}$, will have only one person on his resulting short list.

Since, for $k \in \{1, \dots, q_j/2\}$, ℓ_{2k-1} and ℓ_{2k} are in ρ_{t_k} , $P_{\ell_{2k-1}}$ has $P_{\psi(\ell_{2k-1})}$ on his short-list prior to the elimination, but not subsequently. Similarly, $P_{\ell_{2k}}$ has $P_{\psi(\ell_{2k})}$ on his short-list prior to the elimination, but not subsequently. Thus, the elimination removes $P_{\ell_{2k-1}} = P_{\psi(i_{j-1}+2k-1)}$ from the short list of $P_{i_{j-1}+2k-1} = P_{\psi(\ell_{2k-1})}$ and it removes $P_{\ell_{2k}} = P_{\psi(i_{j-1}+2k)}$ from the short list of $P_{i_{j-1}+2k} = P_{\psi(\ell_{2k})}$.

After eliminating all of the rotations $R_{t_1}, \dots, R_{t_{q_j/2}}$, it follows that the preference list of P_r , for $r \in \rho_j$, will be $Q_{\rho^{-1}r}Q_r$. This means that R_j^d is exposed in the resulting table, showing that R_j has a dual rotation as desired. \square

Example. The dual rotations for the example from Figure 1 are given as follows. $R_1^d = \{E_1^d, H_1^d, S_1^d\}$, $R_2^d = \{E_2^d, H_2^d, S_2^d\}$, $R_3^d = \{E_3^d, H_3^d, S_3^d\}$ and $R_4^d = \{E_4^d, H_4^d, S_4^d\}$. These are given as follows.

$$\begin{aligned} E_1^d &= \{P_2, P_1\}, \\ H_1^d &= \{Q_1, Q_2\}, \\ S_1^d &= \{Q_2, Q_1\}, \\ E_2^d &= \{P_4, P_5, P_6, P_7, P_8, P_1\}, \\ H_2^d &= \{Q_3, Q_4, Q_5, Q_6, Q_7, Q_8\}, \\ S_2^d &= \{Q_4, Q_5, Q_6, Q_7, Q_8, Q_3\}, \\ E_3^d &= \{P_{10}, P_{11}, P_{12}, P_9\}, \\ H_3^d &= \{Q_9, Q_{10}, Q_{11}, Q_{12}\}, \\ S_3^d &= \{Q_{10}, Q_{11}, Q_{12}, Q_9\}, \\ E_4^d &= \{P_{14}, P_{15}, P_{16}, P_{13}\}, \\ H_4^d &= \{Q_{13}, Q_{14}, Q_{15}, Q_{16}\}, \\ S_4^d &= \{Q_{14}, Q_{15}, Q_{16}, Q_{13}\}. \end{aligned}$$

\square

We state one more structural property of the rotation poset, previously proved in [8].

Lemma 13. [8, Lemma 3.5] *If $R = (E, H, S)$ and $R' = (E', H', S')$ are two distinct rotations exposed in a table T , then R removes R' from T if and only*

if $R' = R^d$. Hence, the only way to remove an exposed rotation is to explicitly eliminate it or its dual rotation, if it has one.

Lemma 14. *The only rotations associated with the stable roommate instance I are R_j and R_j^d for $j \in [n]$.*

Proof. Suppose $R = (E, H, S)$ is a rotation different from R_i and R_i^d for all $i \in [n]$, and R is exposed in a table T . Suppose $Q_j \in E$. This, in turn, implies that the preference list of Q_j , which has at least two persons on it, is $P_j P_{\rho j}$. Suppose $j \in \rho_k$. This means that rotation R_k has not been removed and is exposed in table T . Eliminating rotation R would force Q_j to be semi-engaged to $P_{\rho j}$ and remove rotation R_k . From Lemma 13, it follows that the only rotation that could remove the exposed rotation R_k from table T is R_k^d . This implies $R = R_k^d$ which contradicts our assumption that R is different from R_i and R_i^d for $i \in [n]$. Hence, $Q_j \notin E$ for $j \in [2m]$. This establishes that the only persons that could potentially belong to E are P_* persons.

Suppose $P_j \in E$ and the preference list of P_j in table T does not start with $Q_{\rho^{-1}j}$. Let $j \in \rho_k$. This entails that P_j does not belong to the preference list of $Q_{\rho^{-1}j}$. This indicates that rotation R_k has been removed as R_k is exposed at the end of Phase 1. By Lemma 13, it follows that the exposed rotation R_k was removed by eliminating rotation R_k^d . This forces P_j to be semi-engaged to Q_j and P_j has only Q_j on his/her list. This rules out the possibility of $P_j \in E$.

Hence, every $P_l \in E$ has to start with $Q_{\rho^{-1}l}$, i.e. every $P_l \in E$ is semi-engaged to $Q_{\rho^{-1}l}$. The preference list of P_j could read either $Q_{\rho^{-1}j} P_{\psi(j)} Q_j$ or $Q_{\rho^{-1}j} Q_j$. Suppose the preference list of P_j starts with $Q_{\rho^{-1}j} P_{\psi(j)}$. This implies that the next person in the ordered set E is the person semi-engaged to $P_{\psi(j)}$. This is not possible since every person that belongs to E is a P person and every $P_l \in E$ is semi-engaged to $Q_{\rho^{-1}l}$. Therefore, the preference list of every $P_l \in E$ reads $Q_{\rho^{-1}l} Q_l$. Eliminating rotation R from table T would result in every $P_l \in E$ being semi-engaged to Q_l and would remove P_l from $Q_{\rho^{-1}l}$'s list. Suppose $P_i \in E$ and $i \in \rho_k$. This implies that P_i is removed from $Q_{\rho^{-1}i}$'s list and the exposed rotation R_k is removed by rotation R . Again, by Lemma 13, it follows that $R = R_k^d$ which is not possible.

Hence, the only rotations that the stable roommate instance I has are R_j and R_j^d for $j \in [n]$. \square

4.5 Ordering Rotations

In this section we will order the rotations using the *explicitly precedes* relation. We state another lemma from [8].

Lemma 15. [8, Lemma 5.2] *Let p be a person who must be removed from e_i 's list before rotation R is exposed. There exists a unique rotation R' whose elimination removes p from e_i 's list.*

First we establish one fact about the partial order Π^* for our constructed instance I .

Lemma 16. *Rotations R_1, \dots, R_n are minimal elements in the partial order Π^* .*

Proof. We observe that the table obtained at the end of Phase 1 has rotations R_1 through R_n exposed. Hence, there is no rotation R that explicitly precedes R_i for $i \in [n]$. Therefore, R_1 through R_n are minimal elements in the partial order Π^* . \square

Lemma 17. *The only rotations that explicitly precede rotation R_j^d are*

$$\mathcal{R} = \{R_{t_k} : \rho_{t_k} \cap \sigma_j = \{\psi(i_{j-1} + 2k - 1), \psi(i_{j-1} + 2k)\}, 1 \leq k \leq q_j/2\},$$

where σ_j is the j -th σ -cycle $\{\psi(i_{j-1} + 1), \psi(i_{j-1} + 2), \dots, \psi(i_j)\}$ and ρ_{t_k} is the t_k -th ρ -cycle $\{i_{t_k-1} + 1, i_{t_k-1} + 2, \dots, i_{t_k-1} + q_{t_k}\}$ for $1 \leq k \leq q_j/2$.

Proof. In rotation $R_j^d = (E_j^d, H_j^d, S_j^d)$, we have

$$\begin{aligned} E_j^d &= \{P_{i_{j-1}+2}, P_{i_{j-1}+3}, \dots, P_{i_j}, P_{i_{j-1}+1}\}, \\ H_j^d &= \{Q_{i_{j-1}+1}, Q_{i_{j-1}+2}, \dots, Q_{i_j}\}, \\ S_j^d &= \{Q_{i_{j-1}+2}, \dots, Q_{i_j}, Q_{i_{j-1}+1}\}. \end{aligned}$$

The preference lists of members of E_j^d at the end of Phase 1 are as follows for $1 \leq k \leq q_j/2$:

$$\begin{aligned} P_{i_{j-1}+2k-1} : & \quad Q_{\rho^{-1}(i_{j-1}+2k-1)} P_{\psi(i_{j-1}+2k-1)} Q_{i_{j-1}+2k-1}, \\ P_{i_{j-1}+2k} : & \quad Q_{\rho^{-1}(i_{j-1}+2k)} P_{\psi(i_{j-1}+2k)} Q_{i_{j-1}+2k}. \end{aligned}$$

For rotation R_j^d to be exposed, the following has to occur for all $1 \leq k \leq q_j/2$. Person $P_{\psi(i_{j-1}+2k-1)}$ needs to be removed from the list of person $P_{i_{j-1}+2k-1}$ and person $P_{\psi(i_{j-1}+2k)}$ needs to be removed from the list of person $P_{i_{j-1}+2k}$.

Note that after the elimination of R_{t_k} , for $\ell \in \rho_{t_k}$, Q_ℓ is semi-engaged to $P_{\rho\ell}$ and $P_{\rho\ell}$ being semi-engaged to Q_ℓ . At this point, Q_ℓ and $P_{\rho\ell}$ have only one person on their lists. In particular, $P_{\psi(i_{j-1}+2k-1)}$ has only $Q_{\rho^{-1}(\psi(i_{j-1}+2k-1))}$ on his/her list and is removed from $P_{i_{j-1}+2k-1}$'s list. Also, $P_{\psi(i_{j-1}+2k)}$ has only $Q_{\rho^{-1}(\psi(i_{j-1}+2k))}$ on his/her list and is removed from $P_{i_{j-1}+2k}$'s list.

From Lemma 15, it follows that R_{t_k} is the unique rotation that does this job. Therefore, the elimination of rotations $R \in \mathcal{R}$ exposes rotation R_j^d . Since every rotation $R \in \mathcal{R}$ explicitly precedes R_j^d and the elimination of rotations in \mathcal{R} exposes R_j^d , we conclude that the only rotations that explicitly precede rotation R_j^d are $R \in \mathcal{R}$. \square

Example. For the example from Figure 1, the rotations R_1, R_2, R_3, R_4 are minimal elements in the partial order Π^* . The reader can think about these as corresponding to the vertices b_1, b_2, b_3 , and b_4 , respectively, in Figure 1. The only other rotations are R_1^d, R_2^d, R_3^d and R_4^d , which correspond to vertices t_1, t_2, t_3 and t_4 , respectively. From Lemma 17, it can be deduced that R_i explicitly precedes R_j^d if and only if there is an edge from b_i to t_j in the figure. No other rotations explicitly precede R_j^d . \square

4.6 Stocktaking

From Section 4.1, the task was, given the bipartite graph K as described by the ρ and σ cycles, to construct, in polynomial time, a #4-ATTRIBUTE SR-instance I whose nonsingleton rotations can be labelled bijectively with $B \cup T$ so that (G1)–(G4) are satisfied. The construction was done in Sections 4.2 and 4.3. The bijection between rotations and $B \cup T$ is given informally by (6) and (7). More formally, the vertex in B corresponding to the i 'th ρ -cycle is labelled with the rotation R_i and the vertex in T corresponding to the i 'th σ -cycle is labelled with the rotation R_i^d . Lemma 14 guarantees that these are all of the rotations associated with I so the duality relationship gives us (G1). Lemma 17 gives us (G2) and (G4). Lemma 16 gives us (G3).

Thus, we have completed the task, and, by Section 4.1, we have completed the reduction $\#IS \leq_{AP} \#4\text{-ATTRIBUTE SR}$ and the proof that the reduction is correct.

5 Euclidean model

Recall that # k -EUCLIDEAN SR denotes the problem of counting stable assignments for k -Euclidian stable roommate instances. The goal of this section is to prove Theorem 2 which we restate below.

Theorem 2. $\#IS \equiv_{AP} \#k\text{-EUCLIDEAN SR}$ for $k \geq 3$.

Proof. As in the proof of Theorem 1, the main task is to prove $\#IS \leq_{AP} \#3\text{-EUCLIDEAN SR}$. Given an instance of $\#IS$, we show how to construct an

instance of #3-EUCLIDEAN SR whose preference lists are identical to those from Section 4.

We start by assigning position and preference points. We describe the instance of #IS in terms of n ρ -cycles and n σ -cycles as in Section 4.2. As in that section, we have $4m$ people whom we label P_1 through P_{2m} and Q_1 through Q_{2m} . We start by assigning the third co-ordinate of the P_* and the Q_* people. The people Q_1, \dots, Q_{2m} have 0 in their third coordinate. The third coordinate of the position points of P_1, \dots, P_{2m} are arranged on the z -axis, taking each ρ -cycle in order (and leaving a big gap before the next ρ -cycle).

Let $\epsilon = \frac{1}{(2m)^2}$. The i 'th ρ cycle takes up a distance of ϵ (out of a unit distance on the z axis), starting at distance $(i-1)/n$. Let $\theta_i = \epsilon/(7(q_i-1))$. If the i 'th ρ -cycle is the cycle (i_1, \dots, i_d) , then positions $\overline{P}_{i_1}, \dots, \overline{P}_{i_d}$ are assigned in order, leaving a gap of $7\theta_i$ between each pair of people.

More formally, we define the third coordinate of the position points as follows (the asterisks in the first and the second coordinates will be defined shortly):

For $f_i \in \text{Rep}(\rho)$, for $0 \leq k \leq q_i - 1$, set

$$\begin{aligned}\overline{P}_{\rho^k f_i} &= (*, *, (i-1)/n + 7k\theta_i) \quad \text{and} \\ \overline{Q}_{\rho^k f_i} &= (*, *, 0).\end{aligned}$$

We next assign the first and the second coordinates of the position points. These are similar to the last two coordinates in Section 4. These coordinates are arranged around a circle of radius R , where R will be specified later, taking each ρ -cycle in order and leaving big gaps between consecutive ρ -cycles. The i 'th ρ cycle takes up an angle of up to ϵ starting at an angle of $2\pi(i-1)/n$. Let $\theta'_i = \epsilon/4$. If the i 'th ρ -cycle is (i_1, \dots, i_d) , then positions are assigned in the following order:

$$\overline{Q}_{i_1} \overline{P}_{\psi(i_2)} \overline{Q}_{i_2} \overline{P}_{\psi(i_3)} \overline{Q}_{i_3} \overline{P}_{\psi(i_4)} \cdots \overline{Q}_{i_{d-1}} \overline{P}_{\psi(i_d)} \overline{Q}_{i_d} \overline{P}_{\psi(i_1)}.$$

For $k \in \{1, \dots, d-1\}$, the angle between \overline{Q}_{i_k} and $\overline{Q}_{i_{k+1}}$ is $2^{-(k-1)}2\theta'_i$. Also, $\overline{P}_{\psi(i_{k+1})}$ is at an equal angle between these.

To simplify the notation while assigning the i 'th ρ -cycle (i_1, \dots, i_d) , let i_{d+1} denote i_1 . The first two coordinates of the position points are defined in the following manner. Note that the asterisk representing the value of the third coordinate has already been defined above.

For $f_i \in \text{Rep}(\rho)$, for $0 \leq k \leq q_i - 1$, set

$$\begin{aligned}\overline{Q}_{\rho^k f_i} &= \left(R \cos\left(\frac{2\pi(i-1)}{n}\right) + 2\theta'_i \sum_{j=0}^{k-1} 2^{-j}, R \sin\left(\frac{2\pi(i-1)}{n}\right) + 2\theta'_i \sum_{j=0}^{k-1} 2^{-j}, * \right) \text{ and} \\ \overline{P}_{\psi(\rho^{k+1} f_i)} &= \left(R \cos\left(\frac{2\pi(i-1)}{n}\right) + 2\theta'_i \sum_{j=0}^{k-1} 2^{-j} + 2^{-k}\theta'_i, \right. \\ &\quad \left. R \sin\left(\frac{2\pi(i-1)}{n}\right) + 2\theta'_i \sum_{j=0}^{k-1} 2^{-j} + 2^{-k}\theta'_i, * \right).\end{aligned}$$

A sum of the form $\sum_{j=0}^{-1} 2^{-j}$ is taken to be equal to 0. Having defined the position points, we now give the preference points. These are also defined using the ρ -cycles. For the ρ -cycle (i_1, \dots, i_d) , the preference points of $\widehat{Q}_{i_1}, \dots, \widehat{Q}_{i_d}$ are 0 in the first two coordinates. Since every person has the position of his first two coordinates on a circle of radius R centred at the origin, the distance between the preference point of a Q_* person and the position point of another person depends only on the third coordinate. In the third coordinate, for $1 \leq j < d$, \widehat{Q}_{i_j} is placed between \overline{P}_{i_j} and $\overline{P}_{i_{j+1}}$, slightly closer to \overline{P}_{i_j} . Here is the definition. For $f_i \in \text{Rep}(\rho)$, for $0 \leq k \leq q_i - 1$, set

$$\widehat{Q}_{\rho^k f_i} = (0, 0, (i-1)/n + 7k\theta_i + 3\theta_i).$$

The preference points $\widehat{P}_{i_1}, \dots, \widehat{P}_{i_d}$ are 0 in the third coordinate. We note that the contribution of this coordinate towards the distance between its preference point and some position point is at most 1. The preference points of the P people are placed around a circle of radius R in the x - y plane. We pick the radius R large enough, say $10^{10^{4m}}$, to ensure that the third coordinate does not influence the ordering of the initial part of the preference list of a P person. In other words, the only coordinates that affect the order are the x and the y coordinates. Since the x and y coordinates of the positions of the P_* people, the Q_* people and the preference positions of the P_* people all lie on the circle of radius R , the position of a person Y in the initial part of the preference list of a person P_i is purely a function of the angle subtended by the preference point of P_i and the position point of Y at the origin.

If the i 'th ρ -cycle is (i_1, \dots, i_d) , then, in the first two coordinates, for $j > 1$, \widehat{P}_{i_j} is placed $1/3$ of the way along between $\overline{Q}_{i_{j-1}}$ and $\overline{P}_{\psi(i_j)}$. Then, \widehat{P}_{i_1} is placed between \overline{Q}_{i_d} and $\overline{P}_{\psi(i_1)}$, slightly nearer to \overline{Q}_{i_d} . Here is the definition, which is similar to the definition for the last two coordinates in Section 4. For $f_i \in \text{Rep}(\rho)$,

for $0 \leq k \leq q_i - 1$, set

$$\widehat{P}_{\rho^{k+1}f_i} = \left(R \cos\left(\frac{2\pi(i-1)}{n}\right) + 2\theta'_i \sum_{j=0}^{k-1} 2^{-j} + \frac{1}{3}2^{-k}\theta'_i, \right. \\ \left. R \sin\left(\frac{2\pi(i-1)}{n}\right) + 2\theta'_i \sum_{j=0}^{k-1} 2^{-j} + \frac{1}{3}2^{-k}\theta'_i, * \right).$$

It is now easy to check that the preference lists are the same as those in (8) and (9). As in Section 4, there is a slight issue because Q_i is indifferent between the other Q_* people. However, this can be fixed in the same way that it was fixed in the remark at the end of Section 4.3. The rest of the proof is now identical to Section 4. □

6 #1-attribute SR is easy

The goal of this section is to prove Theorem 4 which we restate below.

Theorem 4. *For every #1-ATTRIBUTE SR instance I , there are either 1 or 2 stable assignments. Thus, #1-ATTRIBUTE SR can be solved exactly in polynomial time.*

Proof. We will show that for any #1-ATTRIBUTE SR instance I , there are either 1 or 2 stable assignments. Finding these stable assignments can be done in polynomial time using Gusfield's algorithm [8]. Assume that the instance has n people and, without loss of generality, assume that the positions of these people are ordered $1, \dots, n$. The preference of each person is either “type A”, in which case his preference list is $1, \dots, n$, excluding himself, or his preference is “type B”, in which case his preference list is $n, \dots, 1$, excluding himself. The proof is by induction on n . In the cases below, the notation “ iA ” means that, in instance I , person i has a type-A list (and iB is defined similarly). We start with some base cases.

Base Cases.

(B1) $n = 2$: There is a single stable assignment in which the two people are paired.

(B2) $n = 4$. 1A, 2B, 3B, 4A The lists start out as

$$\begin{array}{c|cccc} 1 & 2 & 3 & 4 & \\ 2 & 4 & 3 & 1 & \\ 3 & 4 & 2 & 1 & \\ 4 & 1 & 2 & 3 & \end{array}$$

Then 2 becomes semi-engaged to 4 so the lists are

$$\begin{array}{c|ccc} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \\ 3 & 2 & 1 & \\ 4 & 1 & 2 & \end{array}$$

Then 3 becomes semi-engaged to 2 so the lists are

$$\begin{array}{c|ccc} 1 & 3 & 4 & \\ 2 & 4 & 3 & \\ 3 & 2 & 1 & \\ 4 & 1 & 2 & \end{array}$$

Now everybody is semi-engaged, so phase 1 ends. We have two exposed rotations,

$$R = \begin{array}{c|cc} 1 & 3 & 4 \\ 2 & 4 & 3 \end{array} \quad R^d = \begin{array}{c|cc} 4 & 1 & 2 \\ 3 & 2 & 1 \end{array}$$

These lead to the two stable assignments $1-4, 2-3$ and $1-3, 2-4$.

- (B3) $n = 4$. **1B, 2A, 3A, 4B** This is symmetric to Case (B2). The symmetry is as follows. Instead of ordering the positions in order $1, 2, 3, 4$ and then, in that order, assigning the lists consistent, inconsistent, inconsistent, consistent, as in Case (B2), think about the backwards order $4, 3, 2, 1$ and assign the lists, in this order, as consistent, inconsistent, inconsistent, consistent.

- (B4) $n = 4$. **1A, 2B, 3A, 4B** The lists start out as

$$\begin{array}{c|ccc} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \\ 3 & 1 & 2 & 4 \\ 4 & 3 & 2 & 1 \end{array}$$

Then 3 becomes semi-engaged to 1 removing pair $(1, 4)$ so the lists are

$$\begin{array}{c|ccc} 1 & 2 & 3 & \\ 2 & 4 & 3 & 1 \\ 3 & 1 & 2 & 4 \\ 4 & 3 & 2 & \end{array}$$

Now everybody is semi-engaged, so phase 1 ends. We have one exposed rotation,

$$R = \begin{array}{c|cc} 1 & 2 & 3 \\ 4 & 3 & 2 \end{array}$$

leading to the unique stable assignment $1-3, 2-4$.

We now give the inductive argument. The proof is easy, apart from checking that all cases are covered. We start by enumerating some cases that can occur, which we call “top inductive step cases” and “mixed inductive step cases”. Then we give some more cases, called “symmetric inductive step cases”. These are symmetric to cases that we’ve already done, so don’t need a new argument. Finally, we conclude with some accounting to check that all cases are covered.

Top Inductive Step Cases (Inductive cases that reduce the number of people, by pairing off two people at the top of the lists)

- (T1) $n \geq 4$. **1A, 2A** Persons 1 and 2 prefer each other, so in phase 1, 1 becomes semi-engaged to 2 and 2 becomes semi-engaged to 1. Then the problem is reduced to an instance of size $n - 2$ without people 1 and 2.
- (T2) $n \geq 4$. **1B, 2A, 3A, i A for some $i > 3$** . Person i becomes semi-engaged to person 1, removing pairs $((1, i - 1), \dots, (1, 2))$. Now persons 2 and 3 prefer each other and can pair off. This gives us an intermediate instance, I' , with two fewer people. I' is not a 1-attribute instance. However, the 1-attribute instance I'' derived from I by removing 2 and 3 has the same stable assignments as I' since it reaches I' by having i semi-engaged to 1.
- (T3) $n \geq 6$. **1A, 2B, 3A, $(n - 1)$ A, n B** Here 3 becomes semi-engaged to 1 removing $(1, 4), \dots, (1, n)$ (including especially $(1, n - 1)$). Then $n - 1$ becomes semi-engaged to 2 removing $(2, 1), \dots, (2, n - 2)$ (including especially $(2, 1)$). Now 1 and 3 can pair off. This gives us an intermediate instance I' , with 2 fewer people. I' is not a 1-attribute instance, so we have to show that the 1-attribute instance I'' derived from I by removing 1 and 3 also gets to I' . This happens by making $n - 1$ semi-engaged to 2 (which is now his first choice).

Mixed Inductive Step Cases

- (M1) $n \geq 4$. **1B, n A** This case reduces to an instance of size $n - 2$ (without people 1 and n) similar to Case (T1).
- (M2) $n \geq 4$. **1A, 2B, i A, n A for some $2 < i < n$** . From instance I , person i becomes semi-engaged to 1. This removes the pairs $(1, i + 1), \dots, (1, n)$ from the lists. Now persons n and 2 prefer each other, and pair off. This gives an instance I' of size $n - 2$ (without people 2 and n). I' is not quite a 1-attribute instance, because the initial semi-engagement of i to 1 knocked out the pairs $(1, i + 1), \dots, (1, n - 1)$ from the lists. However, let I'' be the 1-attribute $(n - 2)$ -person instance derived from instance I by deleting people 2 and n . Note that, from I'' , person i becomes semi-engaged to 1 and then we are

at instance I' . Thus, the stable assignments of I are the stable assignments of I'' .

- (M3) $n \geq 6$. $1\mathbf{A}, 2\mathbf{B}, 3\mathbf{B}, (n-2)\mathbf{A}, (n-1)\mathbf{A}, n\mathbf{B}$ Here 3 becomes semi-engaged to n , removing $(n, 2)$ and $(n, 1)$ so 2 now prefers $n-1$. Then $n-2$ becomes semi-engaged to 1, removing $(1, n-1)$ and $(1, n)$ so $n-1$ now prefers 2. Then 2 and $n-1$ can pair off. Suppose that we started from the original instance I and we just removed people 2 and $n-1$ then we can get to this “paired off” state by having 3 become semi-engaged to n and having $n-2$ become semi-engaged to 1.

Symmetric Inductive Step Cases

- (S1) $n \geq 4$. $(n-1)\mathbf{B}, n\mathbf{B}$ Symmetric to (T1). People $n-1$ and n pair off.
- (S2) $n \geq 4$. $1\mathbf{B}, i\mathbf{B}, (n-1)\mathbf{A}, n\mathbf{B}$ for some $1 < i < n-1$. This is symmetric to Case (M2). The semi-engagement from i to n removes the pairs $(i-1, n), \dots, (1, n)$ so 1 and $n-1$ pair off.
- (S3) $n \geq 4$. $i\mathbf{B}, (n-2)\mathbf{B}, (n-1)\mathbf{B}, n\mathbf{A}$ for some $i < n-2$. Symmetric to case (T2). Person i becomes semi-engaged to person n , removing pairs $((i+1, n), \dots, (n-1, n))$. Now persons $n-2$ and $n-1$ prefer each other and can pair off.
- (S4) $n \geq 6$. $1\mathbf{A}, 2\mathbf{B}, (n-2)\mathbf{B}, (n-1)\mathbf{A}, n\mathbf{B}$ Symmetric to Case (T3). Here $n-2$ becomes semi-engaged to n removing $(1, n), \dots, (n-3, n)$ (including especially $(2, n)$). Then 2 becomes semi-engaged to $n-1$ removing $(3, n-1), \dots, (n, n-1)$ (including especially $(n, n-1)$). Now $n-2$ and n can pair off.

Now let's see that we've covered all cases for $n \geq 4$. Start by looking at all 16 cases for lists 1, 2, $n-1$ and n . Cases (T1), (M1) and (S1) cover all possibilities except

- (C1) $1\mathbf{A} \ 2\mathbf{B} \ (n-1)\mathbf{A} \ n\mathbf{A}$
(C2) $1\mathbf{A} \ 2\mathbf{B} \ (n-1)\mathbf{A} \ n\mathbf{B}$
(C3) $1\mathbf{A} \ 2\mathbf{B} \ (n-1)\mathbf{B} \ n\mathbf{A}$
(C4) $1\mathbf{B} \ 2\mathbf{A} \ (n-1)\mathbf{A} \ n\mathbf{B}$
(C5) $1\mathbf{B} \ 2\mathbf{B} \ (n-1)\mathbf{A} \ n\mathbf{B}$

Also, (C1) is covered by Case (M2) and (C5) is covered by Case (S2). For $n = 4$, (C2), (C3) and (C4) are explicitly covered by our $n = 4$ base cases. For $n = 6$, they are covered as follows.

- (C3): If any of $3, \dots, n - 2$ is an A, then use Case (M2). Otherwise, use Case (S3).
- (C4): Symmetrically, if any of $3, \dots, n - 2$ is a B, then use Case (S2). Otherwise, use Case (T2)
- (C2): This is covered by cases (T3) and (S4) and (M3)

□

7 #BIS hardness of #3-attribute SR and #2-Euclidean SR

7.1 #3-attribute SR

The goal of this section is to prove Theorem 5 which we restate below.

Theorem 5. $\#BIS \leq_{AP} \#3\text{-ATTRIBUTE SR}$.

Proof. We re-use the construction that we used in [3] to reduce #BIS to counting stable assignments in the 3-attribute stable marriage model.

In [3, Section 4.1.1], we show how to take a #BIS instance $G = (V_1 \cup V_2, E)$, where $E \subseteq V_1 \times V_2$ and $|E| = n$ and turn it into a 3-attribute stable matching instance I^* with $3n$ men and $3n$ women which are denoted $\{A_1, \dots, A_n, B_1, \dots, B_n, C_1, \dots, C_n\}$ and $\{a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_n\}$, respectively. We associate two permutations, ρ and σ of $[n]$ with the BIS instance. We show that independent sets of G are in one-to-one correspondence with stable assignments of I^* .

Using the permutations ρ and σ we will now show how to modify the construction to obtain a 3-attribute stable roommate instance I with people

$$\{A_1, \dots, A_n, B_1, \dots, B_n, C_1, \dots, C_n\} \cup \{a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_n\}$$

so that stable assignments for our instance I are in one-to-one correspondence with stable assignments for I^* . Even though the stable roommate instance I simply has $6n$ people (rather than having $3n$ men and $3n$ women, we will refer to the people $\{A_1, \dots, A_n, B_1, \dots, B_n, C_1, \dots, C_n\}$ as “men” and the people $\{a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_n\}$ as “women” to simplify some of the descriptions.

Position and preference vectors for the instance I are defined similarly to those of I^* from [3, Section 4.1.2]. We reproduce here the required notation from [3]: the i 'th σ -cycle has length p_i , the i 'th ρ -cycle has length q_i , $Rep(\rho)$ is the set consisting of one representative for each cycle in ρ , and $Rep(\sigma)$ is the set consisting of one representative for each cycle in σ . The number of σ cycles is ℓ . The number of ρ cycles is k . The only difference between the construction used here and the construction used in [3] is that, instead of using the full unit circle in the first two coordinates to position people, we instead use an angle of $\zeta < \pi/4$ to fit in the position vectors of the women and the preference vectors of the men. The third coordinate of every vector remains unaltered.

$$\text{Let } \epsilon = \frac{\zeta}{n^2}.$$

For $e_i \in Rep(\sigma)$, let $\theta_i = \epsilon/(7p_i - 1)$. Then for $0 \leq m \leq p_i - 1$ define

$$\begin{aligned} \bar{a}_{\sigma^m e_i} &= (\cos(\zeta(i-1)/l + 7m\theta_i + 4\theta_i), \sin(\zeta(i-1)/l + 7m\theta_i + 4\theta_i), 0), \\ \bar{b}_{\rho^m e_i} &= (\cos(\zeta(i-1)/l + 7m\theta_i + 6\theta_i), \sin(\zeta(i-1)/l + 7m\theta_i + 6\theta_i), 4^{\rho^m e_i}), \\ &\text{and} \end{aligned}$$

$$\bar{c}_{\sigma^{m-1} e_i} = (\cos(\zeta(i-1)/l + 7m\theta_i), \sin(\zeta(i-1)/l + 7m\theta_i), 0).$$

$$\text{Let } \phi = 2\pi/100 \text{ and } \epsilon = \frac{\zeta}{n^2}.$$

For $e_i \in Rep(\sigma)$, let $\theta_i = \epsilon/(7p_i - 1)$. Then for $0 \leq m \leq p_i - 1$ define

$$\begin{aligned} \hat{A}_{\sigma^m e_i} &= (\cos(\zeta(i-1)/l + 7m\theta_i + (14/3)\theta_i), \\ &\quad \sin(\zeta(i-1)/l + 7m\theta_i + (14/3)\theta_i), 0), \\ \hat{B}_{\sigma^m e_i} &= (\sin \phi \cos(\zeta(i-1)/l + 7m\theta_i + 4\theta_i), \\ &\quad \sin \phi \sin(\zeta(i-1)/l + 7m\theta_i + 4\theta_i), \cos \phi), \end{aligned}$$

and

$$\hat{C}_{\sigma^{m-1} e_i} = (\cos(\zeta(i-1)/l + 7m\theta_i + (8/5)\theta_i), \sin(\zeta(i-1)/l + 7m\theta_i + (8/5)\theta_i), 0).$$

We note that ϵ and the 2π factor in the cosine and sine terms have been scaled appropriately (relative to the construction in [3]) by the factor $(\zeta/2\pi)$. If we were to restrict the preference lists of the men to the set of women, then these preference lists would match the lists of the men from the stable marriage instance I^* of [3].

Since we are working with a stable roommate instance, we also have to place the position vectors of the the men and the preference vectors of the women in the same 3-dimensional space. We take the position vectors of the men and the preference vectors of the women from the stable marriage instance I^* and modify them as follows:

- (i) We scale down the ϵ and 2π terms in the sine and cosine terms by a factor of $(\zeta/2\pi)$,
- (ii) we offset the angle in the sine and cosine terms by an angle of π , and
- (iii) we negate the third coordinate of all the position and preference vectors.

We summarise the position vectors of the men and the preference vectors of the women in the stable roommate instance I below.

$$\text{Let } \epsilon = \frac{\zeta}{n^2}.$$

For $f_i \in \text{Rep}(\rho)$, let $\omega_i = \epsilon/(7q_i - 1)$. Then for $0 \leq m \leq q_i - 1$ we define

$$\bar{A}_{\rho^{m-1}f_i} = (\cos(\pi + \zeta(i-1)/k + 7m\omega_i), \sin(\pi + \zeta(i-1)/k + 7m\omega_i), 0),$$

$$\bar{B}_{\rho^m f_i} = (\cos(\pi + \zeta(i-1)/k + 7m\omega_i + 4\omega_i), \sin(\pi + \zeta(i-1)/k + 7m\omega_i + 4\omega_i), 0),$$

and

$$\bar{C}_{\rho^m f_i} = (\cos(\pi + \zeta(i-1)/k + 7m\omega_i + 6\omega_i), \sin(\pi + \zeta(i-1)/k + 7m\omega_i + 6\omega_i), -4^{\rho^m f_i}).$$

$$\text{Let } \phi = 2\pi/100 \text{ and } \epsilon = \frac{\zeta}{n^2}.$$

For $f_i \in \text{Rep}(\rho)$, let $\omega_i = \epsilon/(7q_i - 1)$. Then for $0 \leq m \leq q_i - 1$ we define

$$\hat{a}_{\rho^m f_i} = (\sin \phi \cos(\pi + \zeta(i-1)/k + 7m\omega_i + 4\omega_i), \sin \phi \sin(\pi + \zeta(i-1)/k + 7m\omega_i + 4\omega_i), -\cos \phi),$$

$$\hat{b}_{\rho^m f_i} = (\cos(\pi + \zeta(i-1)/k + 7m\omega_i + (8/5)\omega_i), \sin(\pi + \zeta(i-1)/k + 7m\omega_i + 8/5\omega_i), 0), \text{ and}$$

$$\hat{c}_{\rho^m f_i} = (\cos(\pi + \zeta(i-1)/k + 7m\omega_i + 14/3\omega_i), \sin(\pi + \zeta(i-1)/k + 7m\omega_i + (14/3)\omega_i), 0).$$

As observed before, if we were to restrict the preference lists of the women to the set of men then these preference lists would match the lists of the women from the stable marriage instance I^* from [3].

We now state a very simple lemma which we will use to connect our stable roommate instance I to the stable matching instance I^* .

Lemma 18. Suppose vector $\bar{v}_1 = (\cos\theta_1, \sin\theta_1, \alpha_1)$ and vector $\bar{v}_2 = (\cos\theta_2, \sin\theta_2, \alpha_2)$.

(i) If $\pi/2 < |\theta_1 - \theta_2| < 3\pi/2$ and $\alpha_1 \cdot \alpha_2 \leq 0$, then $\bar{v}_1 \cdot \bar{v}_2 < 0$.

(ii) If $0 < |\theta_1 - \theta_2| < \pi/4$ and $\alpha_1 \cdot \alpha_2 \geq 0$, then $\bar{v}_1 \cdot \bar{v}_2 > 0$.

Proof. (i) The dot product $\bar{v}_1 \cdot \bar{v}_2 = \cos(\theta_1 - \theta_2) + \alpha_1 \cdot \alpha_2 < 0$.

(ii) The dot product $\bar{v}_1 \cdot \bar{v}_2 = \cos(\theta_1 - \theta_2) + \alpha_1 \cdot \alpha_2 > 0$. □

Applying this lemma to the preference vectors and the position vectors of the men, we see that the dot product of a preference vector of a man with the position vector of a man is always negative. Similarly, the dot product of a preference vector of a man with the position vector of a woman is always positive. This implies that the initial n positions on the preference lists of the men would be populated by the women and would coincide with the preference lists of the men in the stable matching instance I^* from [3]. The same holds true for the preference lists of the women. The first n positions on their preference lists are occupied by the men and this initial part of their preference lists matches the preference lists of the women from the stable matching instance I^* .

To finish the proof we show that the stable matchings of I are in one-to-one correspondence with the stable matchings of I^* .

First, suppose that M is a stable matching of I^* . It is clear that M is a matching of I , so we must show that it is stable for I . Since M is stable for I^* , it has no man-woman blocking pairs. Also, it is easy to see that, in I , there is no man-man blocking pair (since each man prefers all women to the other men) and similarly, there is no woman-woman blocking pair. Thus, M is a stable matching of I .

Next, suppose that M is a stable matching of I . First, we show that M is a valid matching of I^* — that is, every matched pair consists of one man and one woman. Suppose instead that two men P_i and P_j are matched in I . By the pigeonhole principle, two women p_k and p_ℓ must also be matched in I but now (P_i, p_ℓ) form a blocking pair. Thus, M is a matching of I^* . Since M has no blocking pairs in I it also has no blocking pairs in I^* , so it is a stable matching of I^* .

Thus, the set of stable assignments for the roommate instance I is identical to the set of stable assignments (matchings) for the stable marriage instance I^* . We have already shown in [3] that the latter is in one-to-one correspondence with the independent sets of G , completing the proof. □

7.2 #2-Euclidean SR

The goal of this section is to prove Theorem 6 which we restate below.

Theorem 6. $\#BIS \leq_{AP} \#2\text{-EUCLIDEAN SR}$.

Proof. As in section 7.1, we re-use the construction that we used in [3] to reduce $\#BIS$ to counting stable assignments in the 2-Euclidian stable marriage model.

In [3, Section 6], we show how to take a $\#BIS$ instance $G = (V_1 \cup V_2, E)$, where $E \subseteq V_1 \times V_2$ and $|E| = n$ and turn it into a 2-Euclidian stable matching instance I^* with $3n$ men and $3n$ women which are denoted $\{A_1, \dots, A_n, B_1, \dots, B_n, C_1, \dots, C_n\}$ and $\{a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_n\}$, respectively, as before. Once again, we associate two permutations, ρ and σ of $[n]$ with the BIS instance. We show that independent sets of G are in one-to-one correspondence with stable assignments of I^* .

Using the permutations ρ and σ we will now show how to modify the construction to obtain a 2-Euclidian stable roommate instance I with people

$$\{A_1, \dots, A_n, B_1, \dots, B_n, C_1, \dots, C_n\} \cup \{a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_n\}$$

so that stable assignments for our instance I are in one-to-one correspondence with stable assignments for I^* . We will refer to the people $\{A_1, \dots, A_n, B_1, \dots, B_n, C_1, \dots, C_n\}$ as “men” in I and to the people $\{a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_n\}$ as “women” in order to make it easier to describe the construction.

In the stable roommate construction, the preference points of the men and the position points of the women are the same as those in the 2-Euclidean stable marriage construction from [3]. The preference points of the women are obtained from those in the stable marriage instance by negating both coordinates. Also, the position points of the men are obtained from those in the stable marriage instance by negating both coordinates. Before we define the positions, we wish to remind the reader that the i 'th σ -cycle (out of ℓ σ cycles) has length p_i , the i 'th ρ -cycle (out of k ρ cycles) has length q_i , $Rep(\rho)$ is the set consisting of one representative for each cycle in ρ , and $Rep(\sigma)$ is the set consisting of one representative for each cycle in σ as in [3].

The positions are defined as follows. For $e_j \in Rep(\sigma)$, $f_j \in Rep(\rho)$, $0 \leq h \leq$

$p_j - 1$, and $0 \leq g \leq q_j - 1$ we let

$$\begin{aligned}\bar{a}_{\sigma^h e_j} &= \left(\sum_{i=0}^{j-1} 2p_i + h + 1, 0 \right), \\ \bar{b}_{\rho^g e_j} &= \left(0, \sum_{i=0}^{j-1} 2p_i + h + 1 \right), \\ \bar{c}_{\sigma^{(h-1)} e_j} &= \left(\sum_{i=0}^{j-1} 2p_i + h + 0.3, 0 \right), \\ \bar{A}_{\rho^{g-1} f_j} &= \left(-\sum_{i=0}^{j-1} 2q_i - g - 0.3, 0 \right), \\ \bar{B}_{\rho^g f_j} &= \left(-\sum_{i=0}^{j-1} 2q_i - g - 1, 0 \right), \text{ and} \\ \bar{C}_{\rho^g f_j} &= \left(0, -\sum_{i=0}^{j-1} 2p_i - g - 1 \right).\end{aligned}$$

The preferences are defined as follows. Let $\epsilon = 1/100^n$. For $e_j \in \text{Rep}(\sigma)$, $f_j \in \text{Rep}(\rho)$, $0 \leq h \leq p_j - 1$, $0 \leq g \leq q_j - 1$, we let

$$\begin{aligned}\hat{A}_{\sigma^h e_j} &= \left(\sum_{i=0}^{j-1} 2p_i + h + 1, \sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon \right), \\ \hat{B}_{\sigma^h e_j} &= \left(\sum_{i=0}^{j-1} 2p_i + h + 1, 1000^n \right), \\ \hat{C}_{\sigma^{(h-1)} e_j} &= \left(\sum_{i=0}^{j-1} 2p_i + h + 0.6, 0 \right), \\ \hat{a}_{\rho^g f_j} &= \left(-\sum_{i=0}^{j-1} 2q_i - g - 1, -1000^n \right), \\ \hat{b}_{\rho^g f_j} &= \left(-\sum_{i=0}^{j-1} 2q_i - g - 0.6, 0 \right), \text{ and} \\ \hat{c}_{\rho^g f_j} &= \left(-\sum_{i=0}^{j-1} 2q_i - g - 1, -\sum_{i=0}^{j-1} 2q_i - g - 1 + \epsilon \right).\end{aligned}$$

We show in [3, Section 6] that the preference lists of the stable matching instance I^* have prefixes as described as follows, where τ is a permutation of $[n]$ (we

won't need the details of τ in this paper). In all stable matchings, the men and women of I^* are matched to partners which are included in these prefixes. We refer to these lists as the “initial” preference lists of I^* .

$$\begin{aligned}
& \text{for } f_i \in \text{Rep}(\rho), \\
b_{\rho^m f_i} & : A_{\rho^{(m-1)} f_i} B_{\rho^m f_i} \quad , \quad 0 \leq m \leq q_i - 1, \\
c_{\rho^m f_i} & : B_{\rho^m f_i} C_{\rho^m f_i} \quad , \quad 0 \leq m \leq q_i - 1, \\
a_{\rho^m f_i} & : C_n C_{n-1} \cdots C_1 B_{\rho^m f_i} A_{\rho^m f_i} \quad , \quad 0 \leq m \leq q_i - 2, \text{ and} \\
a_{\rho^{(q_i-1)} f_i} & : C_n C_{n-1} \cdots C_1 \\
& \quad B_{\rho^{(q_i-1)} f_i} A_{\rho^{(q_i-2)} f_i} B_{\rho^{(q_i-2)} f_i} \cdots B_{\rho^2 f_i} A_{\rho f_i} B_{\rho f_i} A_{f_i} B_{f_i} A_{\rho^{(q_i-1)} f_i}.
\end{aligned} \tag{13}$$

$$\begin{aligned}
& \text{For } e_i \in \text{Rep}(\sigma), \\
A_{\sigma^m e_i} & : a_{\sigma^m e_i} b_{\rho \sigma^m e_i} \quad , \quad 0 \leq m \leq p_i - 1, \\
C_{\sigma^{(m-1)} e_i} & : c_{\sigma^{(m-1)} e_i} a_{\sigma^m e_i} \quad , \quad 0 \leq m \leq p_i - 1, \\
B_{\sigma^m e_i} & : b_{\tau(n)} b_{\tau(n-1)} \cdots b_{\tau(1)} a_{\sigma^m e_i} c_{\sigma^m e_i} \quad , \quad 0 \leq m \leq p_i - 2, \text{ and} \\
B_{\sigma^{(p_i-1)} e_i} & : b_{\tau(n)} b_{\tau(n-1)} \cdots b_{\tau(1)} a_{\sigma^{(p_i-1)} e_i} c_{\sigma^{(p_i-2)} e_i} a_{\sigma^{(p_i-2)} e_i} \cdots a_{\sigma e_i} c_{e_i} a_{e_i} c_{\sigma^{(p_i-1)} e_i}.
\end{aligned} \tag{14}$$

We now make three observations. The first of these is self-evident from the construction. We provide justifications below for Observations 2 and 3.

1. For the stable roommate instance I , the preference lists of the men, when restricted to women, match the preference lists from the stable marriage instance I^* . Similarly, the preference lists of the women, when restricted to men, match the preference lists from I^* .
2. For the stable marriage instance I^* , the distance between the preference position of any man and the position point of any woman on his initial preference list is less than the distance between his preference position and the origin. Similarly, the distance between the preference position of any woman and the position point of any man on her initial preference list is less than the distance between her preference position and the origin.
3. For the stable roommate instance I , the distance between the preference position of any man and the origin is less than the distance between his preference position and the position point of any other man. Similarly, the distance between the preference position of any woman and the origin is less than the distance between her preference position and the position point of any other woman.

We now provide arguments that validate Observations 2 and 3. The following calculations use the prefixes of the preference lists of I^* from (13) and (14). To establish Observation 2, we show that the preference point of a man is closer to the last woman on his initial preference list than to the origin. Similarly, we show that the preference point of a woman is closer to the last man on her initial preference list than to the origin.

For man $A_{\sigma^m e_i}$, where $e_i \in \text{Rep}(\sigma)$ and $0 \leq m \leq p_i - 1$, the distance to woman $b_{\rho\sigma^m e_i}$ and the origin ($\bar{0}$) are as follows.

$$\begin{aligned} d^2(\hat{A}_{\sigma^m e_i}, \bar{b}_{\rho\sigma^m e_i}) &= \left(\sum_{j=0}^{i-1} 2p_j + m + 1 - 0 \right)^2 \\ &\quad + \left(\sum_{j=0}^{i-1} 2p_j + m + 1 - \epsilon - \sum_{j=0}^{i-1} 2p_j - m - 1 \right)^2 \\ &= \left(\sum_{j=0}^{i-1} 2p_j + m + 1 \right)^2 + \epsilon^2 \end{aligned}$$

$$\begin{aligned} d^2(\hat{A}_{\sigma^m e_i}, \bar{0}) &= \left(\sum_{j=0}^{i-1} 2p_j + m + 1 - 0 \right)^2 \\ &\quad + \left(\sum_{j=0}^{i-1} 2p_j + m + 1 - \epsilon - 0 \right)^2 \\ &\geq \left(\sum_{j=0}^{i-1} 2p_j + m + 1 \right)^2 + (1 - \epsilon)^2 \\ &> \left(\sum_{j=0}^{i-1} 2p_j + m + 1 \right)^2 + \epsilon^2 = d^2(\hat{A}_{\sigma^m e_i}, \bar{b}_{\rho\sigma^m e_i}) \end{aligned}$$

For man $C_{\sigma^{m-1} e_i}$, where $e_i \in \text{Rep}(\sigma)$ and $0 \leq m \leq p_i - 1$, the distance to woman $a_{\sigma^m e_i}$ and the origin are as follows.

$$\begin{aligned} d^2(\hat{C}_{\sigma^{(m-1)} e_i}, \bar{a}_{\sigma^m e_i}) &= \left(\sum_{j=0}^{i-1} 2p_j + m + 0.6 - \sum_{j=0}^{i-1} 2p_j - m - 1 \right)^2 + (0 - 0)^2 = 0.16. \\ d^2(\hat{C}_{\sigma^{(m-1)} e_i}, \bar{0}) &= \left(\sum_{j=0}^{i-1} 2p_j + m + 0.6 - 0 \right)^2 + (0 - 0)^2 \geq 0.6^2 = 0.36 \\ &> 0.16 = d^2(\hat{C}_{\sigma^{(m-1)} e_i}, \bar{a}_{\sigma^m e_i}) \end{aligned}$$

For man $B_{\sigma^m e_i}$, where $e_i \in \text{Rep}(\sigma)$, we consider two cases: (i) $m \neq p_i - 1$, and (ii) $m = p_i - 1$.

Case(i) $m \neq p_i - 1$: From the preference position of $B_{\sigma^m e_i}$, we compute distances to $c_{\sigma^m e_j}$ and to the origin.

$$\begin{aligned}
d^2(\hat{B}_{\sigma^m e_i}, \bar{c}_{\sigma^m e_i}) &= \left(\sum_{j=0}^{i-1} 2p_j + m + 1 - \sum_{j=0}^{i-1} 2p_j - (m + 1) - 0.3 \right)^2 + (1000^n)^2 \\
&= 0.09 + 1000^{2n} \quad \text{and} \\
d^2(\hat{B}_{\sigma^m e_i}, \bar{0}) &= \left(\sum_{j=0}^{i-1} 2p_j + m + 1 - 0 \right)^2 + (1000^n)^2 \\
&\geq 1 + 1000^{2n} > 0.09 + 1000^{2n} = d^2(\hat{B}_{\sigma^m e_i}, \bar{c}_{\sigma^m e_i}).
\end{aligned}$$

Case(ii) $m = p_i - 1$: From the preference position of $B_{\sigma^m e_i}$, we compute distances to $c_{\sigma^{p_i-1} e_j} = c_{\sigma^{0-1} e_j}$ and to the origin.

$$\begin{aligned}
d^2(\hat{B}_{\sigma^m e_i}, \bar{c}_{\sigma^m e_i}) &= \left(\sum_{j=0}^{i-1} 2p_j + m + 1 - \sum_{j=0}^{i-1} 2p_j - 0.3 \right)^2 + (1000^n)^2 \\
&= (m + 0.7)^2 + 1000^{2n} = (p_i - 1 + 0.7)^2 + 1000^{2n} \\
&= (p_i - 0.3)^2 + 1000^{2n} \quad \text{and} \\
d^2(\hat{B}_{\sigma^m e_i}, \bar{0}) &= \left(\sum_{j=0}^{i-1} 2p_j + m + 1 - 0 \right)^2 + (1000^n)^2 \\
&\geq (m + 1)^2 + 1000^{2n} = (p_i - 1 + 1)^2 + 1000^{2n} \\
&= (p_i)^2 + 1000^{2n} > (p_i - 0.3)^2 + 1000^{2n} = d^2(\hat{B}_{\sigma^m e_i}, \bar{c}_{\sigma^m e_i}).
\end{aligned}$$

The above set of computations and comparisons establish that the preference point of a man is closer to the last woman on his initial preference list than to the origin. We can establish a similar result for the women by repeating the above computations for the preference position of every woman and the position point of the last man on her initial preference list. Hence, we can conclude that Observation 2 holds.

Next we establish Observation 3. We start by comparing the distance between the preference point of any man and the origin with the distance between this preference point and the position point of an A_* or B_* man. We note that the x -coordinate of the position point of an A_* or B_* man is at most -0.3. We also note that the x -coordinate of the preference point of any man is non-negative. In the equations to follow, \hat{X} stands for the preference point of a man and \bar{Y} stands for the position point of an A_* or B_* man. The x and y co-ordinates of \hat{X} will be

denoted \hat{X}_x and \hat{X}_y respectively. The x co-ordinate of \bar{Y} will be denoted \bar{Y}_x . As noted above, $\hat{X}_x \geq 0$ and $\bar{Y}_x \leq -0.3$.

$$\begin{aligned} d^2(\hat{X}, \bar{0}) &= (\hat{X}_x - 0)^2 + (\hat{X}_y - 0)^2 \\ &= (\hat{X}_x)^2 + (\hat{X}_y)^2 \end{aligned}$$

$$\begin{aligned} d^2(\hat{X}, \bar{Y}) &= (\hat{X}_x - \bar{Y}_x)^2 + (\hat{X}_y - 0)^2 \\ &\geq (\hat{X}_x - (-0.3))^2 + (\hat{X}_y - 0)^2 \\ &= (\hat{X}_x + 0.3)^2 + (\hat{X}_y)^2 \\ &> (\hat{X}_x)^2 + (\hat{X}_y)^2 = d^2(\hat{X}, \bar{0}) \end{aligned}$$

Next we compare the distance between the preference point of a man and the origin with the distance between this preference point and the position point of a C_* man. We note that the y -coordinate of the position point of a C_* man is at most -1. We also note that the y -coordinate of the preference point of any man is non-negative. In the equations to follow, \hat{X} stands for the preference point of a man. The x and y co-ordinates of \hat{X} will be denoted \hat{X}_x and \hat{X}_y respectively.

$$\begin{aligned} d^2(\hat{X}, \bar{C}_*) &\geq (\hat{X}_x - 0)^2 + (\hat{X}_y - (-1))^2 \\ &= (\hat{X}_x)^2 + (\hat{X}_y + 1)^2 \end{aligned}$$

$$\begin{aligned} d^2(\hat{X}, \bar{0}) &= (\hat{X}_x - 0)^2 + (\hat{X}_y - 0)^2 \\ &= (\hat{X}_x)^2 + (\hat{X}_y)^2 \\ &< (\hat{X}_x)^2 + (\hat{X}_y + 1)^2 \leq d^2(\hat{X}, \bar{C}_*) \end{aligned}$$

The above calculations establish that the preference point of any man is closer to the origin than the position point of any man. We can establish a similar result for the women. Hence, we can conclude that Observation 3 holds.

Combining Observations 1, 2 and 3, we note that the prefixes of the preference lists of the stable roommate instance I are the same as those of the stable matching instance I^* from (13) and (14). We conclude that the set of stable assignments for the roommate instance I is identical to the set of stable assignments for the stable marriage instance I^* . From [3], we have that the latter is in one-to-one correspondence with the independent sets of G , thereby, establishing the required result.

□

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Appendix 1 — An Example

We give the following example to illustrate the definitions from Section 3 and the two-phase algorithm for finding a stable roommate assignment. Consider the following preference lists.

1	12	7	4	6	9	5	10	2	3	8	11
2	5	6	1	9	12	4	3	10	8	11	7
3	11	9	4	1	8	12	2	6	5	7	10
4	2	9	12	10	7	6	1	8	5	11	3
5	12	6	3	9	4	10	11	8	7	2	1
6	8	4	1	10	2	11	3	5	12	7	9
7	3	5	2	6	10	4	11	1	8	9	12
8	1	7	10	12	3	2	5	4	9	6	11
9	2	12	1	6	5	11	8	10	3	7	4
10	1	4	3	11	2	7	6	8	9	5	12
11	6	4	8	10	12	5	3	1	2	7	9
12	11	6	3	2	7	4	9	10	1	5	8

Phase I proceeds as outlined in Section 3.1, with proposals occurring and “semi-engagements” forming. Here are the short lists at the end of Phase I.

1	7	4	6	9	10						
2	6	9									
3	9	8	12	5							
4	12	10	7	6	1	8	5	11			
5	3	9	4	8	7						
6	8	4	1	10	2						
7	5	10	4	1							
8	10	3	5	4	9	6					
9	2	1	5	8	3						
10	1	4	11	7	6	8					
11	4	10	12								
12	11	3	4								

	3	9	8
	6	8	4
R_1	11	4	10
	8	10	3
	5	3	9

At the end of Phase I, rotation R_1 is exposed in the table (and no other rotations are exposed). Note that R_1 is a singleton rotation. After eliminating this rotation, we have the following table:

1	7	6	9	10	R_2	4	12	10
2	6	9				11	10	12
3	8							
4	12	10	7	6				
5	9	7						
6	4	1	2		R_3	1	7	6
7	5	4	1			2	6	9
8	3					5	9	7
9	2	1	5					
10	1	4	11		R_4	6	4	1
11	10	12				10	1	4
12	11	4						

There are three rotations, R_2, R_3 , and R_4 , exposed in this new table. Using Definition 4, we find that, in the rotation poset for this instance, R_1 precedes each of R_2, R_3 , and R_4 . For example, to see that R_1 explicitly precedes R_2 , take $e_i = 4$ and $\text{ad } p = 2$. Also, rotations R_2, R_3 and R_4 do not precede each other. Each can be performed from the above table.

Also, each of R_2, R_3 , and R_4 has a dual rotation. For example, performing rotation R_4 results in the following table, in which both R_2^d and R_3^d are now exposed (as are R_2 and R_3).

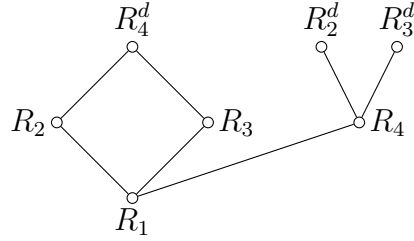
$$\begin{array}{c|cc}
1 & 7 & 6 \\
2 & 6 & 9 \\
3 & 8 & \\
4 & 12 & 10 \\
5 & 9 & 7 \\
6 & 1 & 2 \\
7 & 5 & 1 \\
8 & 3 & \\
9 & 2 & 5 \\
10 & 4 & 11 \\
11 & 10 & 12 \\
12 & 11 & 4
\end{array}
\qquad
\begin{array}{cc|cc}
R_2^d & 10 & 4 & 11 \\
& 12 & 11 & 4
\end{array}
\qquad
\begin{array}{cc|cc}
R_3^d & 6 & 1 & 2 \\
& 9 & 2 & 5 \\
& 7 & 5 & 1
\end{array}$$

So in the rotation poset we have the relations $\Pi^*(R_4, R_2^d)$ and $\Pi^*(R_4, R_3^d)$. Recalling Theorem 9, this also means that $\Pi^*(R_2, R_4^d)$ and $\Pi^*(R_3, R_4^d)$.

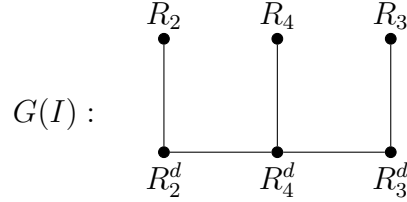
By a careful analysis, we can determine that there are five stable roommate assignments, which we list next along with the set of rotations that leads to each assignment.

Rotations	Stable Assignment
R_1, R_2, R_3, R_4	$(1, 6), (2, 9), (3, 8)$ $(4, 10), (5, 7), (11, 12)$
R_1, R_2, R_3, R_4^d	$(1, 10), (2, 9), (3, 8)$ $(4, 6), (5, 7), (11, 12)$
R_1, R_2, R_4, R_3^d	$(1, 7), (2, 6), (3, 8)$ $(4, 10), (5, 9), (11, 12)$
R_1, R_3, R_4, R_2^d	$(1, 6), (2, 9), (3, 8)$ $(4, 12), (5, 7), (10, 11)$
R_1, R_4, R_2^d, R_3^d	$(1, 7), (2, 6), (3, 8)$ $(4, 12), (5, 9), (10, 11)$

The Hasse diagram of rotation poset of this roommate instance is as follows.



Finally, the graph $G(I)$ (recall the definition from Section 3.2) for this instance is as follows.



Recall that the maximal independent sets in $G(I)$ are in 1-1 correspondence with the stable roommate assignments. These independent sets can be read off directly from the table above using the left-hand column, and deleting R_1 from the set of rotations, e.g. the third assignment $(1, 7), (2, 6), (3, 8), (4, 10), (5, 9), (11, 12)$ corresponds to the maximal independent set $\{R_2, R_4, R_3^d\}$ in $G(I)$.

Appendix 2 — Preference lists and rotations for the example given in Figure 1.

Here are the prefixes (from (8) and (9)) for the example given in Figure 1. First, from the first ρ -cycle, (1, 2) and the first σ -cycle, (3, 4), we have the following lists.

$$\begin{array}{l|l} Q_1 & P_1 P_2 \dots \\ Q_2 & P_2 P_1 \dots \\ P_1 & Q_2 P_3 \{P_4\} Q_1 \dots \\ P_2 & Q_1 P_4 Q_2 \dots \end{array}$$

Then, from the ρ -cycle (3, 4, 5, 6, 7, 8) and the corresponding σ -cycle (1, 2, 9, 10, 13, 14) we have the following.

$$\begin{array}{l|l} Q_3 & P_3 P_4 \dots \\ Q_4 & P_4 P_5 \dots \\ Q_5 & P_5 P_6 \dots \\ Q_6 & P_6 P_7 \dots \\ Q_7 & P_7 P_8 \dots \\ Q_8 & P_8 \{P_7 P_6 P_5 P_4\} P_3 \dots \\ P_3 & Q_8 P_1 \{P_{14} Q_7 P_{13} Q_6 P_{10} Q_5 P_9 Q_4 P_2\} Q_3 \dots \\ P_4 & Q_3 P_2 Q_4 \dots \\ P_5 & Q_4 P_9 Q_5 \dots \\ P_6 & Q_5 P_{10} Q_6 \dots \\ P_7 & Q_6 P_{13} Q_7 \dots \\ P_8 & Q_7 P_{14} Q_8 \dots \end{array}$$

From the ρ -cycle (9, 10, 11, 12) and the corresponding σ -cycle (5, 6, 15, 16) we have the following.

$$\begin{array}{l|l} Q_9 & P_9 P_{10} \dots \\ Q_{10} & P_{10} P_{11} \dots \\ Q_{11} & P_{11} P_{12} \dots \\ Q_{12} & P_{12} \{P_{11} P_{10}\} P_9 \dots \\ P_9 & Q_{12} P_5 \{P_{16} Q_{11} P_{15} Q_{10} P_6\} Q_9 \dots \\ P_{10} & Q_9 P_6 Q_{10} \dots \\ P_{11} & Q_{10} P_{15} Q_{11} \dots \\ P_{12} & Q_{11} P_{16} Q_{12} \dots \end{array}$$

Similarly, from the ρ -cycle (13, 14, 15, 16) and the corresponding σ -cycle (7, 8, 11, 12) we have the following.

$$\begin{array}{l|l}
Q_{13} & P_{13}P_{14} \cdots \\
Q_{14} & P_{14}P_{15} \cdots \\
Q_{15} & P_{15}P_{16} \cdots \\
Q_{16} & P_{16}\{P_{15}P_{14}\}P_{13} \cdots \\
P_{13} & Q_{16}P_7\{P_{12}Q_{15}P_{11}Q_{14}P_8\}Q_{13} \cdots \\
P_{14} & Q_{13}P_8Q_{14} \cdots \\
P_{15} & Q_{14}P_{11}Q_{15} \cdots \\
P_{16} & Q_{15}P_{12}Q_{16} \cdots
\end{array}$$

The short lists (from Equations (11) and (12)) are therefore as follows.

$$\begin{array}{l|l}
Q_1 & P_1P_2 \cdots \\
Q_2 & P_2P_1 \cdots \\
P_1 & Q_2P_3Q_1 \cdots \\
P_2 & Q_1P_4Q_2 \cdots
\end{array}$$

$$\begin{array}{l|l}
Q_3 & P_3P_4 \cdots \\
Q_4 & P_4P_5 \cdots \\
Q_5 & P_5P_6 \cdots \\
Q_6 & P_6P_7 \cdots \\
Q_7 & P_7P_8 \cdots \\
Q_8 & P_8P_3 \cdots \\
P_3 & Q_8P_1Q_3 \cdots \\
P_4 & Q_3P_2Q_4 \cdots \\
P_5 & Q_4P_9Q_5 \cdots \\
P_6 & Q_5P_{10}Q_6 \cdots \\
P_7 & Q_6P_{13}Q_7 \cdots \\
P_8 & Q_7P_{14}Q_8 \cdots
\end{array}$$

$$\begin{array}{l|l}
Q_9 & P_9P_{10} \cdots \\
Q_{10} & P_{10}P_{11} \cdots \\
Q_{11} & P_{11}P_{12} \cdots \\
Q_{12} & P_{12}P_9 \cdots \\
P_9 & Q_{12}P_5Q_9 \cdots \\
P_{10} & Q_9P_6Q_{10} \cdots \\
P_{11} & Q_{10}P_{15}Q_{11} \cdots \\
P_{12} & Q_{11}P_{16}Q_{12} \cdots
\end{array}$$

$$\begin{array}{l|l}
Q_{13} & P_{13}P_{14} \cdots \\
Q_{14} & P_{14}P_{15} \cdots \\
Q_{15} & P_{15}P_{16} \cdots \\
Q_{16} & P_{16}P_{13} \cdots \\
P_{13} & Q_{16}P_7Q_{13} \cdots \\
P_{14} & Q_{13}P_8Q_{14} \cdots \\
P_{15} & Q_{14}P_{11}Q_{15} \cdots \\
P_{16} & Q_{15}P_{12}Q_{16} \cdots
\end{array}$$